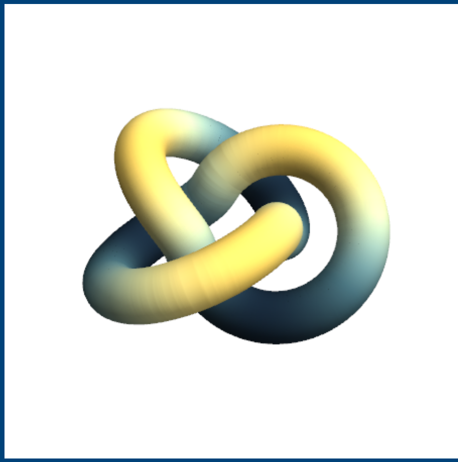
Evidence for the Bible



Volume One: Physics and Cosmology

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Evidence for the Bible, Volume One: Physics and Cosmology
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Chapter 1

What We Can Know And How We Can Know It

Here we will look at the Bible to determine what we can know and how. Once we have done this, we can see how much physics comes out. Apparently, everything comes out!

Consider Proverbs 2:1-8,

Proverbs 2:1. My son, if thou wilt receive my words, and hide my commandments with thee;
Proverbs 2:2. So that thou incline thine ear unto wisdom, and apply thine heart to understanding;
Proverbs 2:3. Yea, if thou criest after knowledge, and liftest up thy voice for understanding;
Proverbs 2:4. If thou seekest her as silver, and searchest for her as for hid treasures;
Proverbs 2:5. Then shalt thou understand the fear of the Lord, and find the knowledge of God.
Proverbs 2:6. For the Lord giveth wisdom: out of his mouth cometh knowledge and understanding.
Proverbs 2:7. He layeth up sound wisdom for the righteous: he is a buckler to them that walk uprightly.
Proverbs 2:8. He keepeth the paths of judgment, and preserveth the way of his saints.

We see that if we apply ourselves to learning, understanding and internalizing God's Law, we shall:

1. Understand the fear of The Lord (i.e. understand His character).

2. Find the knowledge of God. If this simply meant knowledge of God in isolation, then there is no point in adding this statement as an addition to the previous statement. This verse therefore must also mean that one will obtain knowledge which comes from God, that God has seen fit to let us be able to acquire.

3. Notice that the search for knowledge is likened to a hunt for hid treasures. Evidently there is knowledge available from God, the nature of which is not obvious at a surface level, and which only becomes apparent by deep internalizing of God's Law and character (of which His Law is a transcript).

4. Receive wisdom which is sound, and receive knowledge and understanding.

Question: Does this knowledge and understanding only apply to an awareness of right and wrong, or is it more universal, encompassing all of God's handiwork? Consider Psalms 19:1-8,

Psalms 19:1. The heavens declare the glory of God; and the firmament sheweth his handywork.
Psalms 19:2. Day unto day uttereth speech, and night unto night sheweth knowledge.
Psalms 19:3. There is no speech nor language, where their voice is not heard.
Psalms 19:4. Their line is gone out through all the earth, and their words to the end of the world. In them hath he set a tabernacle for the sun,
Psalms 19:5. Which is as a bridegroom coming out of his chamber, and rejoiceth as a strong man to run a race.
Psalms 19:6. His going forth is from the end of the heaven, and his circuit unto the ends of it: and there is nothing hid from the heat thereof.
Psalms 19:7. The law of the Lord is perfect, converting the soul: the testimony of the Lord is sure, making wise the simple.
Psalms 19:8. The statutes of the Lord are right, rejoicing the heart: the commandment of the Lord is pure, enlightening the eyes.

We see that:

1. The physical heavens are declaring the glory of God while showing His handywork. The physical heavens would not be able to declare God's character (i.e. God's glory) unless His Law was encoded and observable in the Heavens. It should be evident therefore that if one understands God's Law, one will also be able to understand the physical heavens. Hence we see that understanding God's Law is foundational to understanding the laws of physics. The laws of physics must be derivable from God's great Law, the transcript of His character; the perfect reflection of Who He is.

2. The physical heavens are declaring God's glory night after night. Hence it is something intrinsic to the heavens in general that is communicating this information about God's glory. Not just a one off supernatural or natural sign somewhere.

3. The sun is set in the physical heavens so we know we have correctly identified the heavens as referring to at least the observable universe.

4. The testimony of The Lord, His Law, makes wise the simple. The direct context of this is the heavens night after night showing us knowledge of God's glory. It follows that an understanding of the universe can be obtained by the ignorant, by applying God's Law.

5. The commandment of The Lord enlightens the eyes. An implication of this is that without God's Law, ones eyes are not enlightened. The obvious implication of this is that the laws

of physics must be derivable from God's Law, and, at a deeper level, understandable by a knowledge of Who God is, since that is ultimately what His Law is telling us. Since we have seen that the laws of physics are derivable from God's Law, it follows that the laws of physics flow from an understanding of who God is.

A successful attempt at understanding physics and the universe needs to rest on God's Law, and, ultimately, on God Himself. Concerning God Himself, here is some Biblical data about both God and His Son:

1. Jesus has a God (John 20:17),
2. The God of Jesus is The God of us (John 20:17),
3. The God of Jesus is His Father (John 20:17),
4. The God of Jesus is also our Father (John 20:17),
5. Jesus was with God, even before creation (John 17:5),
6. Jesus is God's only begotten Son (John 1:18 and matthew 3:16-17),
7. Jesus is called the Word of God (John 1:1 together with John 1:14),
8. Jesus, being the Word of God, is God (John 1:1),
9. The Father is "the only true God" and has the executive primacy. Jesus executes His Fathers will (John 17:3, Luke 2:49),
10. Jesus is equal with God by virtue of God being His Father (John 5:18),
11. God's Name is in His Son (obviously by virtue of the fact that Jesus is God's Son) (Exodus 23:21, Isaiah 9:6),
12. The "oneness" of The Father and The Son is the same as the oneness between God's disciples. God's disciples are of one accord (Acts 1:14) and are all composed of the same stuff. Jesus is "composed" of the same stuff as His Father (Jesus is God; John 1:1) and is in total cooperative agreement with His Father (see point 9).
13. Jesus is known as the wisdom of God (1 Corinthians 1:24).
    item Jesus is the express image of His Father (Hebrews 1:3; see further below).
14. **The character of The Father and the character of Jesus are one and the same** (John 14:9).
15. God created everything and created everything through His Son (John 1:3, Hebrews 1:2).
16. God continually actively upholds and sustains creations existence, and does so through His Son (Colossians 1:17).

Here is some character related data about God:

1 John 4:8. He that loveth not knoweth not God; for God is love."

The fundamental core of God's character is love. But love has different out workings. Consider 1 Corinthians 13:4-5 which describes various actions that arise from the principle of love (in the King James Version of the Bible the word "charity" is used),

1 Corinthians 13:4. Charity suffereth long, and is kind; charity envieth not; charity vaunteth not itself, is not puffed up,
1 Corinthians 13:5. Doth not behave itself unseemly, seeketh not her own, is not easily provoked, thinketh no evil;

One might now wonder if there is a particular outworking of love which is more fundamental, or whether all of the listed qualities are equally elementary, that is so say, underived from any of the others. Following on from 1 John 4:8, consider what is said directly after, in 1 John 4:9-11,

1 John 4:9. In this was manifested the love of God toward us, because that God sent his only begotten Son into the world, that we might live through him.
1 John 4:10. Herein is love, not that we loved God, but that he loved us, and sent his Son to be the propitiation for our sins.
1 John 4:11. Beloved, if God so loved us, we ought also to love one another.

So we see that God's character fundamentally is that of continuously giving out when it comes to God's intelligent creation. One might wonder how this active principle of love seeking not its own worked before creation. The answer is that God already had someone to love; His only begotten Son Jesus. We now examine just Who this Jesus is, especially as He relates to His Father.

Chapter 2

God, His Law And The Resulting Laws Of Physics; The Theory Of Everything

The most direct way of applying the instructions in the Bible concerning how to extract an understanding about physics and the universe, is to look into God's Law, and examine the activities of God and His Son in light of His Law. We will find that if one does this, the theory of everything falls out automatically. Here is God's Law:

Matthew 22:37. Jesus said unto him, Thou shalt love the Lord thy God with all thy heart, and with all thy soul, and with all thy mind.
Matthew 22:38. This is the first and great commandment.
Matthew 22:39. And the second is like unto it, Thou shalt love thy neighbor as thyself.
Matthew 22:40. On these two commandments hang all the law and the prophets.

Jesus is quoting from Deuteronomy 6:5 and Leviticus 19:18,

Deuteronomy 6:5. And thou shalt love the Lord thy God with all thine heart, and with all thy soul, and with all thy might.
Leviticus 19:18. Thou shalt not avenge, nor bear any grudge against the children of thy people, but thou shalt love thy neighbor as thyself: I am the Lord.

Recall that all knowledge comes ultimately from internalizing God's Law. If we are going to internalize these commandments we need to understand why God gave these commandments. Remember that God created all things in harmony with His character which has love as its foundation. Love is "giving centered," or, to put it another way, "others centered." We know God's law is the way it is because it needs to be that way. It's there as a result of God loving His creation and having its best interests in mind. If the commandments were broken then something would go wrong. The first great commandment has two classes: God is in one of the classes, and all of creation is in the other class. What is the primary relationship between these classes?

God as Creator caused the other class to exist. All must be in harmony with God by virtue of how God created all in the first place, which means that everything, in order to reflect God, needs to reflect God's loving (i.e. "giving") causality that is in harmony with God's giving-causal character. The underlying principle in the first great commandment which necessitates the commandment being what it is, is what one might choose to call the Causality Principle. If we abstract the principle from this commandment we could state the Causality Principle as follows:

Causality Principle: The Causality Principle asserts that whenever something happens between two entities, no matter how trivial or nontrivial, something results, no matter how trivial or nontrivial.

We will see later that this principle is not so trivial as to be useless. It will turn out to be the very singular key needed to unlock a vast amount of physics. Regarding the mathematical encoding of this principle, we note that causality can be represented by a diagram of arrows meeting tail to head. Unevaluated algebraic expressions have this sort of graphical representation, which, if nondegenerate (which must be the case when one considers the commandment to "love thy neighbor as thyself"), requires a normed division algebra. There are only four elementary normed division algebras:

1. The algebra over the real numbers $\mathbb{R}$,
2. The algebra over the complex numbers $\mathbb{C}$,
3. The algebra over the quaternion numbers $\mathbb{H}$,
4. The algebra over the octonion numbers $\mathbb{O}$,

which are the first four algebras $\mathbb{A}_0$, $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, of an infinite sequence of Cayley-Dickson algebras $\mathbb{A}_m$. The largest of these algebras that is entirely central is $\mathbb{C}$ so whatever our state space is, it will be a vector space over $\mathbb{C}$. Classical physics information is encoded in $\mathbb{R}$, quantum physics information is encoded in $\mathbb{C}$, spacetime information is encoded in $\mathbb{H}$ and particle content is encoded in $\mathbb{O}$. Interactions and all that results from them, is built up on these four normed division algebras. For more on causal sets in physics, see for example, [9, 13, 17, 59].

Since Jesus said that both of the two great commandments are what all of the law and the prophets hang on, we had better look at the second great commandment, abstract the basic principle from the commandment, and then take the principles together as a basis upon which to derive physics. For convenience, we also reproduce the second great commandment here:

Matthew 22:39. And the second is like unto it, Thou shalt love thy neighbor as thyself.

If all loved their neighbor, then there exists reciprocity. There is a duality here. Consider two people; person A and person B. Suppose person A loved person B. Let us symbolically represent that as A(B). Now suppose instead that person B loved person A. Let us symbolically represent that as B(A). Let us further label the lover person as the "observer" and let us label the loved person as the "state." In the scenario represented as A(B), A is the observer and B is the state. In the other scenario represented as B(A), B is the observer and A is the state. The abstract principle under girding the second great commandment is that, by the commandment, God asserts[1] the rule that A(B) and B(A) be both a reality, rather than only one of those statements being a reality. The injection of reciprocity makes the two statements become equivalent. And if equivalent, then there is a duality in the sense that it matters not whether one writes A(B) or B(A) because both now imply the same universal reality that one might represent symbolically as A $\leftrightarrow$ B.

In other words, by abstracting the principle which under girds the second great commandment, we see a symmetry principle between the observer and the state. We might choose to call this principle the Observer-State Symmetry Principle. We could state the Observer-State Symmetry Principle as follows:

Observer-State Symmetry Principle: Let us denote a set of observers by A and a set of states by B. The action of the observers on the states gives some set of physics outcomes. The Observer-State Symmetry Principle asserts that if one reverses the labels and arrows of implication, and labels A as the set of states and B as the set of observers, then the action of B on A must give the same physics.

The Observer-State Symmetry Principle was first proposed as a fundamental requirement for the semantic structure of any ultimately correct theory of physics by Majid. More details about the principle can be found in [56]. Original presentations of the principle can be found in [54, 55, 53] and a more recent discussion can be found in [57]. We have derived two foundational symmetry principles for physics from God's Law. What we lack now is the space of physical states upon which to implement these two symmetry principles. That is fine, because we have not yet looked at the relationship between God and His Son in light of His Law. The relationship between God and His Son Jesus embodies the outworking of God's Law. We will thus need to mathematically encode the relationship between God and His Son. This will give us the space of physical states. This, together with the imposition of the two previously derived symmetry principles will then be the theory of everything. Simple as that. All that remains from that point onwards is to unpack the theory of everything to prove that what we have is indeed the theory of everything.

The first step in building the picture is to represent the total space of actions among The Father and The Son abstractly by $\mathcal{H}$. God and His Son are independently self-existent since Jesus has His Father's original unborrowed underived life intrinsically within Himself by virtue of His Sonship. Hence we can represent the abstract space of possible actions by God as $\mathcal{H}_G$ and we can represent the abstract space of possible actions by His Son as $\mathcal{H}_S$. At this stage, the total space of actions can then be written as $\mathcal{H} = \mathcal{H}_G \otimes \mathcal{H}_S$. Now let a basis for $\mathcal{H}_G$ be denoted by $\{|s\rangle_G\}$ and a basis for $\mathcal{H}_S$ be denoted by $\{|\psi\rangle_S\}$. If the actions of God and His Son were independent from each other (which is not actually the case), a general action state $|\psi\rangle \in \mathcal{H}$ would be of the form

$$|\psi\rangle = \sum_{i,j} z_{ij} \, |s_i\rangle_G \otimes |\psi_j\rangle_S \, . \tag{2.1}$$

where $z_{ij} = z_i z_j$ to reflect the independence of the two subspaces $\mathcal{H}_G$ and $\mathcal{H}_S$.

However the actions of God and His Son are **not** independent of each other. Neither are Their two actions random. There is a consistent permanent relationship between the two actions

[1]based on God's infinite wisdom in recognizing that for the good of creation, it has to be this way. Hence God asserts it. Remember that God's commandments are not arbitrary. He does not just make up random commandments that might not necessarily have a deep necessity to them and that would fail to be a huge blessing.

which allows us to write down a well-defined **entangled** state. Anything God does, He does through His Son, and moreover, anything done by The Son is done only to carry out the will of His Father. Hence mathematically one can think of $\mathcal{H}_S$ as the space of all possible actions (which may be infinite-dimensional), and we may think of $\mathcal{H}_G$ as the two-dimensional space spanned by two states corresponding to "yes" or "no," or, else one could think of it as "is" and "is not." Without this entanglement dichotomy no nontrivial structure could exist.

The most basic physical action that could possibly be taken would then take the form of the following quantum entangled state in $\mathcal{H} = \mathcal{H}_G \otimes \mathcal{H}_S$:

$$|\psi\rangle = z_0 \, |0\rangle_{G_1} \otimes |\psi_0\rangle_{S_{G_1}} + z_1 \, |1\rangle_{G_1} \otimes |\psi_1\rangle_{S_{G_1}} , \qquad z_0, z_1 \in \mathbb{C}. \tag{2.2}$$

By virtue of the consistency of the relationship between God and His Son, we can define sub actions within each action along the same lines and hence a division process emerges with can be pursued indefinitely. Also, given that the state $|\psi\rangle$ by definition gives all that is, in terms of the actions carried out by God and His Son, then in terms of physics we can identify $|\psi\rangle$ as being the physical state representing the universe.

We thus arrive at the theory of everything:

1. **Structure: A quantum theory.**
2. **Symmetry Principle: The non-existence of a preferred basis on the state space.**
3. **Configuration Space: The Universe: Considered as a single inseparable quantum entangled state $|\psi\rangle \in \mathcal{H}$.**

It may not be immediately obvious why we would call this the theory of everything. Hence the next chapter is devoted to unpacking this structure to see what is there. We will find that it **uniquely** contains what we see in physics. These physical starting tenets for describing the universe were first discussed, as far as we are aware, in [10], to which we refer the reader for more details.

Chapter 3

Unpacking The Theory Of Everything

3.1 Introduction

The point of this chapter is to unpack the theory of everything presented in the previous chapter, which directly results from considering the relationship between God and His Son in light of God's Law, which is a transcript of God's character. This chapter basically amounts to a demonstration of the truths of God's Word, by showing its claims to be true. We have followed the Biblical instruction and it has resulted in a vast amount of understanding, much of it of which will be unpacked and show cased in this chapter. Since all that follows in this chapter is built on following the Bible, the chapter constitutes a proof that physics and the Bible are in perfect harmony with each other.

As explained in [10], this iteration process for representing the universe in terms of some basis forms a foam $\mathcal{F}^\infty$ of binary reference points. To see what physics emerges from all of this we need to understand the topological structure of $\mathcal{H}$. Firstly, the universe is represented by a single inseparable self-interacting entangled quantum state $|\psi\rangle \in \mathcal{H}$ defined up to a $U(1)$ phase. Hence the topology of $\mathcal{H}$ is S^1. What about the topological structure of subdivisions? For any $k \geq 1$ corresponding to a number of reference points within, such as $\mathcal{F}^k = \{G_1, \cdots, G_k\}$ the universe can be represented as a sum of correlated outcomes labeled by bits sequences s of length k:

$$\mathcal{H}_{G_1} \otimes \cdots \otimes \mathcal{H}_{G_k} \otimes \mathcal{H}_{S_{G_1 \cdots G_k}} \ni |\psi\rangle = \sum_{s \in \mathcal{F}^k} z_s \, |s\rangle_{G_1 \cdots G_k} \otimes |\psi_s\rangle_{S_{G_1 \cdots G_k}}, \qquad (\langle s|s'\rangle = \delta_{ss'}). \quad (3.1)$$

All of the physics comes from the interference phenomena which are described by the coherence factors $\langle \psi_s|\psi_{s'}\rangle$. Unitary equivalence of all states implies that the topology of the partitioned Hilbert space $\mathcal{H}_{G_1} \otimes \cdots \otimes \mathcal{H}_{G_k} \otimes \mathcal{H}_{S_{G_1 \cdots G_k}}$ has to be identified with $S^{2^{k+1}-1}$. The sum over s contains 2^k terms with complex coefficients.

Two consecutive ranks of representations following a division step are related by

$$|\psi_s\rangle_{S_{G_1 \cdots G_k}} = z_{s|0} \, |0\rangle_{G_{k+1}} \otimes \left|\psi_{s|0}\right\rangle_{S_{G_1 \cdots G_{k+1}}} + z_{s|1} \, |1\rangle_{G_{k+1}} \otimes \left|\psi_{s|1}\right\rangle_{S_{G_1 \cdots G_{k+1}}} \quad (3.2)$$

where $s|s'$ represents two bits sequences concatenation.

Topological considerations are the basis for studying the Hilbert space automorphism. The division process relates to the inductive doubling process that leads to the infinite family of Cayley-Dickson algebras A_k. The hypermatrix description of quantum propositions in $[\mathbb{C}^2]^k$ can be replaced with a matrix description with values in A_k which has dimension 2^k on $\mathbb{R}$. Here the topology of the Hilbert space $\mathcal{H}_{G_1} \otimes \cdots \otimes \mathcal{H}_{G_k} \otimes \mathcal{H}_{S_{G_1 \cdots G_k}}$ may be identified as

$$S^1_{A_k} = \{(\alpha, \beta) \in A_k^2 | \alpha^2 + \beta^2 = 1\}. \tag{3.3}$$

The important thing here is that the consecutive partitioned Hilbert spaces present a nested bundle structure that is mapped to an **octonionic structure**:

$$\mathcal{H}_{\mathcal{F}^{k-3}} \otimes \mathcal{H}_{S_{\mathcal{F}^{k-3}}} \hookrightarrow \mathcal{H}_{\mathcal{F}^{k-2}} \otimes \mathcal{H}_{S_{\mathcal{F}^{k-2}}} \hookrightarrow \mathcal{H}_{\mathcal{F}^{k-1}} \otimes \mathcal{H}_{S_{\mathcal{F}^{k-1}}} \hookrightarrow \mathcal{H}_{\mathcal{F}^{k}} \otimes \mathcal{H}_{S_{\mathcal{F}^{k}}} \tag{3.4}$$

$$S^1_{A_{k-3}} \hookrightarrow S^1_{A_{k-2}} \hookrightarrow S^1_{A_{k-1}} \hookrightarrow S^1_{A_k}$$

$$S^1_{A_{k-2}} \to A_{k-2}P^1, \quad S^1_{A_{k-1}} \to A_{k-1}P^1, \quad S^1_{A_k} \to A_k P^1.$$

We can refer to this structure as a universal fractal. The first part of the sequence explicitly is

$$S^1_{\mathbb{R}} \hookrightarrow S^1_{\mathbb{C}} \hookrightarrow S^1_{\mathbb{H}} \hookrightarrow S^1_{\mathbb{O}}, \tag{3.5}$$

$$S^1_{\mathbb{C}} \to \mathbb{C}P^1, \quad S^1_{\mathbb{H}} \to \mathbb{H}P^1, \quad S^1_{\mathbb{O}} \to \mathbb{O}P^1.$$

which, as we shall see, has exactly the needed structure to explain the particles of the standard model along with their strong, weak and electromagnetic forces and also the emergence of 3+1-dimensional spacetime and gravity.

The higher Cayley-Dickson algebras such as the sedenions $\mathbb{S}$ and the trigintaduonions $\mathbb{T}$ e.t.c. have zero divisors. The way to proceed is by employing the theory of **varying complexity representations of Cayley-Dickson algebras**. That is, for any $\alpha \in A_k$ have the unique decomposition

$$\begin{aligned} \alpha &= \sum_{i=0}^{2^k-1} \xi_i \mathbf{e}_i \\ &= \sum_{j=0}^{2^{k-m}-1} \left(\sum_{l=0}^{2^m-1} \xi_{2^m j + l} \mathbf{e}_l \right) \mathbf{e}_{2^m j} \\ &= \sum_{j=0}^{2^{k-m}-1} \beta_j(\alpha) \mathbf{e}_{2^m j} \end{aligned} \tag{3.6}$$

with $\{\mathbf{e}_i\}$ the canonical basis of A_k and $\beta_j(\alpha) \in A_m$. The structure only depends on the multiplication table in A_k where $\mathbf{e}_{2^m j + l} = \mathbf{e}_l \mathbf{e}_{2^m j}$.

We now consider the global algebra of nonlocal observables that emerges from this structure.

The fundamental representation of A_k with maximal complexity is obtained using coefficients in the largest alternative division algebra, the **octonions** by setting $m = 3$ so that $\beta_j(\alpha) \in \mathbb{O}$. One has to use the octonions in order to divide. This fact combined with the structure of the universal fractal indicate that the fundamental observables of the Hilbert space $\mathcal{H}_\mathcal{F} \otimes \mathcal{H}_{S_\mathcal{F}}$ can be represented by elements of the exceptional Jordan algebra $J_3^\mathbb{O}$, the states of which correspond to points in the octonionic projective plane $\mathbb{O}P^2$. This is also the only object where one can simultaneously have the three Hopf fibrations to $\mathbb{O}P^1$ as lines in $\mathbb{O}P^2$. Also, $J_3^\mathbb{O}$ elements satisfy the needed algebraic conditions to represent projection operators associated to quantum mechanical states. Moreover, $J_3^\mathbb{O}$ produces Chern-Simons-like phases, making it compatible with loop quantum gravity.

Only invariant quantities have a physical meaning. There is a unique trilinear form

$$(\cdot,\cdot,\cdot) : J_3^\mathbb{O} \times J_3^\mathbb{O} \times J_3^\mathbb{O} \to \mathbb{R} \tag{3.7}$$

such that for $A \in J_3^\mathbb{O}$ we have $(A, A, A) = \det(A)$. This determinant is invariant under $E_6 = SL(3, \mathbb{O})$ transformations.

The diagonal of all $J_3^\mathbb{O}$ matrices are preserved by $SO(8) \subset E_6$ or its double cover $Spin(8)$. $Spin(8)$ has the nonlinear triality outer automorphism which acts to permute the vector, left-handed spinor, and right-handed spinor representations. Since triality is relative to three distinct references, it is non-local. Hence going local involves triality breaking.

We now focus on the quantum algebra of local observables.

In the universal fractal we have

$$\mathrm{Aut}(\mathbb{C}P^1) = SL(2, \mathbb{C}), \quad \mathrm{Aut}(\mathbb{H}P^1) = SL(2, \mathbb{H}), \quad \mathrm{Aut}(\mathbb{O}P^1) = SL(2, \mathbb{O}). \tag{3.8}$$

If one then isolates the entanglement contributions to the local references $SL(2, \mathbb{O})$ and $SL(2, \mathbb{H})$ we get further taken to $SL(2, \mathbb{C})$.

To describe particle physics, which is local, one must break $E_6 = SL(3, \mathbb{O})$ to either $SL(2, \mathbb{O})$, $SL(2, \mathbb{H})$ or $SL(2, \mathbb{C})$.

Locality breaks the triality symmetry. We need to understand the triality symmetry within the algebra of local quantum observables. The three possible fibrations to the octonionic projective line in the universal fractal is related to the fact that there exist three distinct ways in which $J_2^\mathbb{O}$ embeds into $J_3^\mathbb{O}$. Hence there are three distinct overlapping copies of $SL(2, \mathbb{O})$ in $SL(3, \mathbb{O})$. However there is only one $SO(8)$ transformation subgroup of the three copies of $SL(2, \mathbb{O})$ in an $E_6 = SL(3, \mathbb{O})$ transformation. Hence the fibrations down to the local structure break the triality symmetry. Hence locally we get spinors and vectors, the interactions of which is in terms of the action of $J_2^\mathbb{O}$ on $\mathbb{O}^2$ by matrix multiplication.

We have seen that unpacking what we claim to be the theory of everything results in us needing to fully elucidate the structure of the exceptional Jordan algebra $J_3^\mathbb{O}$. We then need to consider how multiple copies of $J_3^\mathbb{O}$ interact. However, before we even look at one copy of $J_3^\mathbb{O}$ we should first look at the structure in $\mathbb{H}$ and then look at the structure contained in $\mathbb{O}$ and fully understand what physics is present at those levels first. Moreover since $\mathbb{C}$ is the largest algebra among the Cayley-Dickson algebras which is central, we consider the complexifications $\mathbb{C} \otimes \mathbb{H}$ and $\mathbb{C} \otimes \mathbb{O}$.

3.2 The physics in the algebra of complex quaternions $\mathbb{C} \otimes \mathbb{H}$

A general element $q \in \mathbb{C} \otimes \mathbb{H}$, in the standard basis, takes the form

$$q = q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{C}, \quad \mathbf{e}_i \in \mathbb{H} \tag{3.9}$$

where $\mathbf{e}_i^2 = -1$, $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ and $i, j = 1, 2, 3$. As vector spaces $\mathbb{C} \otimes \mathbb{H} = \mathbb{C} \oplus \mathbb{C}^3$. Moreover the automorphism groups are $\mathrm{Aut}(\mathbb{H}) \cong SO(3)$ and $\mathrm{Aut}(\mathbb{C} \otimes \mathbb{H}) = SL(2, \mathbb{C})$ which is the cover of the Lorentz group. We also have three different conjugations q^*, $\tilde{q}$ and $q^\dagger$ respectively called the complex conjugate, the quaternion conjugate and the hermitian quaternion conjugate, respectively defined as:

$$* : i \mapsto -i, \quad \tilde{} : \mathbf{e}_i \mapsto -\mathbf{e}_i, \quad \dagger \equiv * \circ \tilde{} : i \mapsto -i, \quad \mathbf{e}_i \mapsto -\mathbf{e}_i \tag{3.10}$$

where i, not to be confused with subscripts labeled by the same symbol, is the unit imaginary element $i \in \mathbb{C}$. So far, it all looks like pure maths with no apparent physics. Since we have causality being given by unevaluated algebraic expressions, and since the corresponding arrow-based graphs have algebraic multiplication associated with propagation along an arrow and addition associated with the joining of arrows, it follows that physically well-defined entities that are stable under propagation must correspond to algebraic elements that are preserved under multiplication. Mathematically this means that stable physical states must be identifiable with the minimal left/right ideals of the algebra. However, before proceeding, we recall that we arrived at what is called the Observer-State Symmetry Principle. In terms of matrix representations of an algebra what this implies is that we do not want to identify column vectors v with states but rather matrices $vv^\dagger$ with states where here in the matrix context the dagger implies transposition in addition to hermitian conjugate. But a square matrix can be both left multiplied and right multiplied. The Observer-State Symmetry Principle therefore implies that it is not the algebra itself we are physically interested in, but rather the endomorphisms of the algebra. So we actually want to look at the minimal left/right ideals of $\mathrm{End}(\mathbb{C} \otimes \mathbb{H})$. There is a duality between looking at minimal left ideals or minimal right ideals. They present two different ways of encoding the same physics so we are free to pick one to work with as a convention. Hence we will choose the minimal left ideals as the convention.

Finding the minimal left ideals in terms of Clifford algebras is explained in [22, 52]. For more on these sorts of issues, see [1, 16, 24, 25]. In what follows in this section and in the next section, we follow [22].

To find the minimal left ideals of $\mathrm{End}(\mathbb{C} \otimes \mathbb{H})$ we note that $\mathrm{End}(\mathbb{C} \otimes \mathbb{H}) \cong \mathbb{C}\ell(4)$, where $\mathbb{C}\ell(4) \equiv \mathbb{C} \otimes C\ell(p, q)$, $(p + q = 4,\ \mathbb{Z} \ni p, q \geq 0)$ is the $2^4 = 16$ complex-dimensional (32-real-dimensional) complexified Clifford algebra generated by $\mathbf{e}_i$, $i = 1, 2, 3, 4$ with $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$, and, due to the complexification it does not matter how many generators square to $+1$ and how many generators square to -1. Note that if we were to have ignored the Observer-State Symmetry Principle, we would have been thinking about the minimal left ideals of $\mathbb{C} \otimes \mathbb{H} \cong \mathbb{C}\ell(2)$. The quaternions have inequivalence between left and right multiplication. Hence the endomorphism algebra is not $\mathbb{C}\ell(2)$ but rather $\mathbb{C}\ell(2) \otimes_{\mathbb{C}} \mathbb{C}\ell(2) \cong \mathbb{C}\ell(4)$. Nevertheless, to build up properly so we can see what structure emerges from each building block, we will start off by looking at $\mathbb{C}\ell(2) \cong C\ell(3, 0)$ where $C\ell(3, 0)$ is the real Clifford algebra where all three generators square to $+1$ with zero generators squaring to -1.

The Clifford algebra $\mathbb{C}\ell(2)$ has standard basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12} \equiv \mathbf{e}_1\mathbf{e}_2\}$. To write down the minimal left ideals we:

1. Identify a vector space V spanned by the elements $\{\mathbf{e}_1, \mathbf{e}_2\}$ over $\mathbb{C}$ with quadratic form being the anticommutator $Q(v) = \{v, v\}$.

2. The maximally totally isotropic subspace U of V in this case is one-complex-dimensional, spanned by $\alpha_1 \equiv \frac{1}{2}(-i\mathbf{e}_2 + \mathbf{e}_1)$. To get the second minimal left ideal, one at this point writes down $\frac{1}{2}(i\mathbf{e}_2 + \mathbf{e}_1)$ instead.

3. By virtue of U being only one-dimensional, the nilpotent object is simply $\Omega = \alpha_1$.

4. The primitive idempotent is then $\Omega\Omega^\dagger = \alpha_1\alpha_1^\dagger = \frac{1}{2}(1 - i\mathbf{e}_{12})$.

5. This particular minimal left ideal is given by $\Psi_R = \mathbb{C}\otimes\mathbb{H}\Omega\Omega^\dagger = \Psi_R^+\alpha_1^\dagger\Omega\Omega^\dagger + \Psi_R^-\Omega\Omega^\dagger$ where $\Psi_R^\pm \in \mathbb{C}\otimes\mathbb{H}$ and $\alpha_1^\dagger\Omega\Omega^\dagger = -\frac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2)$. The second minimal left ideal can be obtained via the starting point mentioned in step 2 above, or equivalently by simply complex conjugating to get $\Psi_L = \Psi_L^+\Omega^*\tilde{\Omega} + \Psi_L^-\tilde{\alpha}_1\Omega^*\tilde{\Omega}$, $\Psi_L^\pm \in \mathbb{C}\otimes\mathbb{H}$.

If we transform Ψ_L and Ψ_R by elements of $\mathrm{Aut}(\mathbb{C}\otimes\mathbb{H}) = SL(2,\mathbb{C})$ we find that Ψ_L transforms as the left-handed Weyl spinor and Ψ_R transforms as the right-handed Weyl spinor. Moreover, the sum $\Psi_D \equiv \Psi_L + \Psi_R$ transforms as a Dirac spinor under $\mathbb{C}\ell(4)$ with the γ matrices being replaced by

$$\gamma^0 \mapsto 1|\mathbf{e}_1, \quad \gamma^1 \mapsto \mathbf{e}_1| - i\mathbf{e}_2, \quad \gamma^2 \mapsto \mathbf{e}_2| - i\mathbf{e}_2, \quad \gamma^3 \mapsto -i\mathbf{e}_{12}| - i\mathbf{e}_2, \quad \gamma^5 \mapsto -1| - i\mathbf{e}_{12}. \tag{3.11}$$

Here $a|b$ means the operation consisting of multiplying on the left by a and multiplying on the right by b. Four-vectors can be constructed with $p_\mu \in \mathbb{R}$ by $\hat{p} \equiv p_\mu\gamma^\mu = p_0 1|\mathbf{e}_1 + p_j\mathbf{e}_j| - i\mathbf{e}_2$, $\mu = 0,1,2,3$. Parity transformations are given by $\gamma^0 \mapsto \gamma^0 = 1|\mathbf{e}_1$ and $\gamma^j \mapsto -\gamma^j = -\mathbf{e}_j| - i\mathbf{e}_2$. An antisymmetric rank 2 tensor can be formed as $\hat{F} \equiv F_{\mu\nu}\gamma^{\mu\nu}$ where $F_{\mu\nu} \in \mathbf{R}$ and $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. We also see that $\gamma^0\hat{p}\Psi_l = p^*\Psi_L$, $\gamma^0\hat{p}\Psi_R = p\Psi_R$, $\hat{F}\Psi_L = -F^*\Psi_L$ and $\hat{F}\Psi_R = -F\Psi_R$.

We have seen in this section that the spacetime Lorentzian spin degrees of freedom emerge from the complex quaternionic structure. We now turn to the complex octonions.

3.3 The physics in the algebra of complex octonions $\mathbb{C}\otimes\mathbb{O}$

In the standard basis an octonion takes the form

$$a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4 + a_5\mathbf{e}_5 + a_6\mathbf{e}_6 + a_7\mathbf{e}_7. \tag{3.12}$$

We also have $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ for all $(i \neq j)$ and $\mathbf{e}_i^2 = -1$, $i,j = 1,\cdots,7$. The multiplication rules are $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_4$ and cylic, $\mathbf{e}_1\mathbf{e}_5 = \mathbf{e}_6$ and cyclic, $\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_7$ and cyclic, $\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_5$ and cyclic, $\mathbf{e}_2\mathbf{e}_6 = \mathbf{e}_7$ and cyclic, $\mathbf{e}_3\mathbf{e}_4 = \mathbf{e}_6$ and cyclic, $\mathbf{e}_4\mathbf{e}_5 = \mathbf{e}_7$ and cyclic. This can all be represented diagramatically by a Fano plane. This is important particularly because it describes the octonionic projective plane $\mathbb{O}P^2$ that serves as the state space for the exceptional Jordan algebra $J_3^{\mathbb{O}}$ which we look at in detail later. The octonions are nonassociative. However there are associative quaternionic triples contained in the octonions. In the standard basis these are given by each class of multiplication rules already listed so for example the first associative quaternionic triple is the set $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4)$. As in the previous section, we are really actually interested in $\mathrm{End}(\mathbb{C}\otimes\mathbb{O})$. These maps, being maps, are associative. Moreover any right multiplication of an octonion by an octonion can always be re-expressed as some kind of left multiplication (in general quite complicated and hard to find) so we have the isomorphism $\mathrm{End}(\mathbb{C}\otimes\mathbb{O})\cong \mathbb{C}\ell(6)$ which is a $2^6=64$-complex-dimensional algebra.

To find the minimal left ideals:

1. Let V be the complex vector space spanned by $\{i\mathbf{e}_1, i\mathbf{e}_2, \cdots, i\mathbf{e}_6\}$ with $\mathbf{e}_j^2=-1$, $j=1,\cdots,6$. The quadratic form is again $Q(v)=\{v,v\}$.
2. Define $U\subset V$ by the spanning set $\alpha_1=\frac{1}{2}(-\mathbf{e}_5+i\mathbf{e}_4)$, $\alpha_2=\frac{1}{2}(-\mathbf{e}_3+i\mathbf{e}_1)$ and $\alpha_3=\frac{1}{2}(-\mathbf{e}_6+i\mathbf{e}_2)$. These satisfy, for arbitrary $v\in\mathbb{C}\otimes\mathbb{O}$, $\{\alpha_i,\alpha_j\}v=0$, $\{\alpha_i^\dagger,\alpha_j^\dagger\}v=0$, $\{\alpha_i^*,\alpha_j^*\}v=0$, $\{\tilde\alpha_i,\tilde\alpha_j\}v=0$, $\{\alpha_i,\alpha_j^\dagger\}v=\delta_{ij}v$, $\{\alpha_i^*,\tilde\alpha_j\}v=\delta_{ij}v$.
3. The nilpotent is then $\Omega\equiv\alpha_1\alpha_2\alpha_3$ and the corresponding primitive idempotent is $\Omega\Omega^\dagger=\alpha_1\alpha_2\alpha_3\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$.
4. The minimal left ideal defined by this is $S^u\equiv\mathrm{End}(\mathbb{C}\otimes\mathbb{O})\Omega\Omega^\dagger$. The other minimal left ideal, S^d, is based on α_i^* and $\tilde\alpha_i$ and is the complex conjugate of S^u.

To find the symmetries of the maximally totally isotropic subspace U, we form the hermitian operators of the form ${\alpha'}^\dagger\alpha+\alpha^\dagger\alpha'$ where $\alpha\equiv c_1\alpha_1+c_2\alpha_2+c_3\alpha_3$ and $\alpha'\equiv c_1'\alpha_1+c_2'\alpha_2+c_3'\alpha_3$ with $c_i, c_j'\in\mathbb{C}$. Expanding everything out gives, for real r_0 and r_i,

$$r_0Q+\sum_{i=1}^{8}r_i\Lambda_i \tag{3.13}$$

where $Q\equiv\frac{1}{3}(\alpha_1^\dagger\alpha_1+\alpha_2^\dagger\alpha_2+\alpha_3^\dagger\alpha_3)$ is the electromagnetic charge operator which generates $U(1)$ and

$$\begin{aligned}
\Lambda_1 &= -\alpha_2^\dagger\alpha_1-\alpha_1^\dagger\alpha_2, & \Lambda_2 &= i\alpha_2^\dagger\alpha_1-i\alpha_1^\dagger\alpha_2,\\
\Lambda_3 &= \alpha_2^\dagger\alpha_2-\alpha_1^\dagger\alpha_1, & \Lambda_4 &= -\alpha_1^\dagger\alpha_3-\alpha_3^\dagger\alpha_1,\\
\Lambda_5 &= -i\alpha_1^\dagger\alpha_3-i\alpha_3^\dagger\alpha_1, & \Lambda_6 &= -\alpha_3^\dagger\alpha_2-\alpha_2^\dagger\alpha_3,\\
\Lambda_7 &= -i\alpha_3^\dagger\alpha_2-i\alpha_2^\dagger\alpha_3, & \Lambda_8 &= -\frac{1}{\sqrt{3}}\left(\alpha_1^\dagger\alpha_1+\alpha_2^\dagger\alpha_2-2\alpha_3^\dagger\alpha_3\right)
\end{aligned}\tag{3.14}$$

are the generators of $SU(3)$, satisfying the usual $su(3)$ commutators. The transformations of the α and $\alpha^\dagger$ are both of the form

$$e^{i(r_0Q+\sum_{i=1}^8 r_i\Lambda_i)}ae^{-i(r_0Q+\sum_{i=1}^8 r_i\Lambda_i)}. \tag{3.15}$$

For the minimal left ideal S^u we have

$$e^{i(r_0Q+\sum_{i=1}^8 r_i\Lambda_i)}S^u e^{-i(r_0Q+\sum_{i=1}^8 r_i\Lambda_i)}=e^{i(r_0Q+\sum_{i=1}^8 r_i\Lambda_i)}S^u \tag{3.16}$$

since $\Omega^\dagger\alpha_i^\dagger=0$, $i=1,2,3$. By transforming each of the eight elements of S^u as shown above and each of the eight elements of S^d by the conjugate representation, we find that the electromagnetic and color charge quantum numbers with each element allow us to write

$$\begin{aligned}
S^u &= \bar\nu\Omega\Omega^\dagger+\bar d^r\alpha_1^\dagger\Omega\Omega^\dagger+\bar d^g\alpha_2^\dagger\Omega\Omega^\dagger+\bar d^b\alpha_3^\dagger\Omega\Omega^\dagger\\
&+ u^r\alpha_3^\dagger\alpha_2^\dagger\Omega\Omega^\dagger+u^g\alpha_1^\dagger\alpha_3^\dagger\Omega\Omega^\dagger+u^b\alpha_2^\dagger\alpha_1^\dagger\Omega\Omega^\dagger+e^+\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\Omega\Omega^\dagger
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned} S^d &= \nu\Omega^\dagger\Omega + d^r\alpha_1\Omega^\dagger\Omega + d^g\alpha_2\Omega^\dagger\Omega + d^b\alpha_3\Omega^\dagger\Omega \\ &+ \bar{u}^{\bar{r}}\alpha_3\alpha_2\Omega^\dagger\Omega + \bar{u}^{\bar{g}}\alpha_1\alpha_3\Omega^\dagger\Omega + \bar{u}^{\bar{b}}\alpha_2\alpha_1\Omega^\dagger\Omega + e^-\alpha_3\alpha_2\alpha_1\Omega^\dagger\Omega \end{aligned} \tag{3.18}$$

where all coefficients are complex and are labeled for easy identification with the first generation of elementary fermions. Notice that S^u consists of all of the weak isospin up states while S^d consists of all of the weak isospin down states.

One might correctly argue that the identification of each element of the minimal left ideals with the elementary first generation standard model fermions is premature because we have not yet shown that the elements have the correct weak force quantum numbers, nor have we shown what the gravitational properties of the states are. To pursue both of these ends and more, we carry on.

3.4 The Dirac algebra, the weak force and $\mathbb{C}\ell(6)$

The point of this section, the contents of which can be found in an appendix of [68], is to show how the weak interactions result in an extension of the Dirac algebra $\mathbb{C}\ell(4)$ to $\mathbb{C}\ell(6)$. This then sets up the next section, which provides a full identification of one full generation of the standard model fermions, apart from looking at their gravitational properties. We deal with (quantum) gravity later.

The Dirac algebra $\mathbb{C}\ell(4)$ can be thought of as the even subalgebra $\mathbb{C}\ell^+(5)$ of $\mathbb{C}\ell(5)$. Likewise, the algebra $\mathbb{C}\ell(5)$ can be thought of as the even subalgebra $\mathbb{C}\ell^+(6)$ of $\mathbb{C}\ell(6)$. The extension of the Dirac algebra $\mathbb{C}\ell^+(5)$ to $\mathbb{C}\ell^+(6)$ is brought about by the inclusion of the weak isospin generator T_3 and the extension from $\mathbb{C}\ell^+(6)$ to $\mathbb{C}\ell(6)$ is brought about by the inclusion of the other two weak generators T_1 and T_2. So the extension can be considered in two steps.

Firstly, we write $\Gamma^0 \equiv 1_2 \otimes \gamma^0$ where 1_2 is the 2×2 identity matrix. Next we define $\tau^j \equiv 2T_j + 2T_j\Gamma^0 = \sigma_j \otimes 1_4$. Next we denote the weak isospin-up subspace by $\mathbb{W}_0$ and the weak isospin down subspace by $\mathbb{W}_1$. Then $\Gamma^\mu = 1_2 \otimes \gamma^\mu \in \mathrm{End}_{\mathbb{C}}(\mathbb{W}_0 \oplus \mathbb{W}_1)$ and the elements $\omega^j \equiv \tau^j\Gamma^5$ anticommute with Γ^μ and satisfy $\omega^j\omega^k = \delta_{jk} + i\epsilon^{jkl}\tau^l$, $j,k,l = 1,2,3$. The elements $\omega^3\Gamma^\mu$ and ω^3 form a basis which generates $\mathbb{C}\ell^+(6) \cong \mathbb{C}\ell(5)$. A basis which generates the full $\mathbb{C}\ell(6)$ is given by $\omega^1\Gamma^\mu$, ω^1 and τ^2.

3.5 One generation of standard model fermions from $\mathbb{C}\ell(6)$

In this section we follow [68]. We take the Clifford algebra $\mathbb{C}\ell(6) \equiv \bigoplus_{k=0}^{3}(\bigwedge^\circ \chi^\dagger)\mathfrak{p}\bigwedge^k \chi$, show how the matter particles of nature are actually naturally present in this algebra and show why they have the properties they do and feel the forces that they do. This section covers everything concerning the defining properties of the first generation of standard model fermions except for quantum gravity which will be covered later.

A basis for the left ideal $\bigwedge^\circ \chi^\dagger\mathfrak{p} \cong \mathrm{Mat}(8,\mathbb{C})$ is

$$(1\mathfrak{p}, q_{23}^\dagger\mathfrak{p}, q_{31}^\dagger\mathfrak{p}, q_{12}^\dagger\mathfrak{p}, q_{321}^\dagger\mathfrak{p}, q_1^\dagger\mathfrak{p}, q_2^\dagger\mathfrak{p}, q_3^\dagger\mathfrak{p}). \tag{3.19}$$

Each basis element in order can be thought of as an 8×8 matrix with a nonzero entry somewhere in the first column with all other entries in the matrix being zero. The first basis element corresponds to a matrix with a 1 in the first row and first column. The second basis element corresponds to a matrix with a nonzero number in the second row, first column with all other entries zero and so on. Some of the entries are $+1$ while others are -1 but that is of no physical consequence and is just a pure convention choice of representation.

A basis for the right ideal $\bigwedge^{\circ} \chi$ is

$$(1\mathfrak{p}, \mathfrak{p}q_{23}, \mathfrak{p}q_{31}, \mathfrak{p}q_{12}, \mathfrak{p}q_{321}, \mathfrak{p}q_1, \mathfrak{p}q_2, \mathfrak{p}q_3). \tag{3.20}$$

This can be interpreted analogously to the above except here we can think of a set of eight 8×8 matrices where all entries are zero except for single entries in the first row progressing from left to right as one looks at each successive basis element. Again some nonzero matrix entries are $+1$ while others are -1.

Using these two bases, we can construct the entire 8×8 matrix A which gets acted on from the left and from the right by $\mathbb{C}\ell(6)$. Symbolically, this matrix of physical states has the form:

$$A = \begin{pmatrix} 1\mathfrak{p}1 & 1\mathfrak{p}q_{23} & 1\mathfrak{p}q_{31} & 1\mathfrak{p}q_{12} & 1\mathfrak{p}q_{321} & 1\mathfrak{p}q_1 & 1\mathfrak{p}q_2 & 1\mathfrak{p}q_3 \\ q_{23}^\dagger\mathfrak{p}1 & q_{23}^\dagger\mathfrak{p}q_{23} & q_{23}^\dagger\mathfrak{p}q_{31} & q_{23}^\dagger\mathfrak{p}q_{12} & q_{23}^\dagger\mathfrak{p}q_{321} & q_{23}^\dagger\mathfrak{p}q_1 & q_{23}^\dagger\mathfrak{p}q_2 & q_{23}^\dagger\mathfrak{p}q_3 \\ q_{31}^\dagger\mathfrak{p}1 & q_{31}^\dagger\mathfrak{p}q_{23} & q_{31}^\dagger\mathfrak{p}q_{31} & q_{31}^\dagger\mathfrak{p}q_{12} & q_{31}^\dagger\mathfrak{p}q_{321} & q_{31}^\dagger\mathfrak{p}q_1 & q_{31}^\dagger\mathfrak{p}q_2 & q_{31}^\dagger\mathfrak{p}q_3 \\ q_{12}^\dagger\mathfrak{p}1 & q_{12}^\dagger\mathfrak{p}q_{23} & q_{12}^\dagger\mathfrak{p}q_{31} & q_{12}^\dagger\mathfrak{p}q_{12} & q_{12}^\dagger\mathfrak{p}q_{321} & q_{12}^\dagger\mathfrak{p}q_1 & q_{12}^\dagger\mathfrak{p}q_2 & q_{12}^\dagger\mathfrak{p}q_3 \\ q_{321}^\dagger\mathfrak{p}1 & q_{321}^\dagger\mathfrak{p}q_{23} & q_{321}^\dagger\mathfrak{p}q_{31} & q_{321}^\dagger\mathfrak{p}q_{12} & q_{321}^\dagger\mathfrak{p}q_{321} & q_{321}^\dagger\mathfrak{p}q_1 & q_{321}^\dagger\mathfrak{p}q_2 & q_{321}^\dagger\mathfrak{p}q_3 \\ q_1^\dagger\mathfrak{p}1 & q_1^\dagger\mathfrak{p}q_{23} & q_1^\dagger\mathfrak{p}q_{31} & q_1^\dagger\mathfrak{p}q_{12} & q_1^\dagger\mathfrak{p}q_{321} & q_1^\dagger\mathfrak{p}q_1 & q_1^\dagger\mathfrak{p}q_2 & q_1^\dagger\mathfrak{p}q_3 \\ q_2^\dagger\mathfrak{p}1 & q_2^\dagger\mathfrak{p}q_{23} & q_2^\dagger\mathfrak{p}q_{31} & q_2^\dagger\mathfrak{p}q_{12} & q_2^\dagger\mathfrak{p}q_{321} & q_2^\dagger\mathfrak{p}q_1 & q_2^\dagger\mathfrak{p}q_2 & q_2^\dagger\mathfrak{p}q_3 \\ q_3^\dagger\mathfrak{p}1 & q_3^\dagger\mathfrak{p}q_{23} & q_3^\dagger\mathfrak{p}q_{31} & q_3^\dagger\mathfrak{p}q_{12} & q_3^\dagger\mathfrak{p}q_{321} & q_3^\dagger\mathfrak{p}q_1 & q_3^\dagger\mathfrak{p}q_2 & q_3^\dagger\mathfrak{p}q_3 \end{pmatrix}. \tag{3.21}$$

Each element of this matrix corresponds to an elementary physical state. Our task is to identify them with nature. We now give a concrete choice of matrix representations to facilitate concrete calculations that are needed in order to make the matter particle state identifications. The particular basis for the matrix representations are given as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.22}$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_3^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.23}$$

$$q_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ -i\sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & \sigma_3^- \\ 0 & 0 & -\sigma_3^- & 0 \\ 0 & -\sigma_3^+ & 0 & 0 \\ \sigma_3^+ & 0 & 0 & 0 \end{pmatrix}, \tag{3.24}$$

$$q_3^\dagger = \begin{pmatrix} 0 & 0 & 0 & -\sigma_- \\ 0 & 0 & \sigma_+ & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ \sigma_- & 0 & 0 & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \end{pmatrix}, \tag{3.25}$$

$$q_2 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3^+ \\ 0 & 0 & -\sigma_3^+ & 0 \\ 0 & -\sigma_3^- & 0 & 0 \\ \sigma_3^- & 0 & 0 & 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & 0 & 0 & \sigma_+ \\ 0 & 0 & -\sigma_- & 0 \\ 0 & \sigma_- & 0 & 0 \\ -\sigma_+ & 0 & 0 & 0 \end{pmatrix}, \tag{3.26}$$

$$\mathfrak{q} = q_{123}, \quad \mathfrak{q}^\dagger = q_3^\dagger q_2^\dagger q_1^\dagger, \quad \mathfrak{p} = \mathfrak{q}\mathfrak{q}^\dagger, \quad \mathfrak{p}' = \mathfrak{q}^\dagger \mathfrak{q}. \tag{3.27}$$

Moreover the elements q_i, $q_j^\dagger$ satisfy

$$\{q_i, q_j\} = 0, \quad \{q_i^\dagger, q_j^\dagger\} = 0, \quad \{q_i, q_j^\dagger\} = \delta_{ij}, \quad \mathfrak{q}^2 = \mathfrak{q}^{\dagger 2} = 0. \tag{3.28}$$

The elements $\mathfrak{q}$ and $\mathfrak{q}^\dagger$ are therefore nilpotent whereas $\mathfrak{p}$ and $\mathfrak{p}'$ are idempotent: that is $\mathfrak{p}^2 = \mathfrak{p}$, $(\mathfrak{p}')^2 = \mathfrak{p}'$. They are also primitive so they define respectively minimal left and right ideals of the algebra $\mathbb{C}\ell(6) \cong \mathrm{End}(\mathbb{C} \otimes \mathbb{O})$.

3.5.1 The weak force $SU(2)_L$

By defining the elements $\mathfrak{e}_i = \mathfrak{q} + \mathfrak{q}_i^\dagger$ and $\tilde{\mathfrak{e}}_i = i(\mathfrak{q}_i^\dagger - \mathfrak{q}_i)$ we can form the elements

$$T_1 = \frac{-i}{4}(-\tilde{\mathfrak{e}}_1\mathfrak{e}_1\mathfrak{e}_2 - i\mathfrak{e}_2), \quad T_2 = \frac{-i}{4}(\mathfrak{e}_3\tilde{\mathfrak{e}}_2\mathfrak{e}_2 + i\tilde{\mathfrak{e}}\mathfrak{e}_3\mathfrak{e}_1), \quad T_3 = \frac{-i}{4}(i\mathfrak{e}_2\mathfrak{e}_3\tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3 + \tilde{\mathfrak{e}}\mathfrak{e}) \tag{3.29}$$

which satisfy $[T_i, T_j] = i\epsilon_{ijk}T_k$. Consider the left ideal

$$S_1 = (a_1 + a_2 q_{23}^\dagger + a_3 q_{31}^\dagger + a_4 q_{12}^\dagger + a_5 q_{321}^\dagger + a_6 q_1^\dagger + a_7 q_2^\dagger + a_8 q_3^\dagger)\mathfrak{p}. \tag{3.30}$$

The matrix representation turns out as

$$S_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.31}$$

We see that

$$T_1 S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.32}$$

$$T_2 S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ia_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -ia_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ia_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ia_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.33}$$

and

$$T_3 S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.34}$$

We see that exact same pattern when we left multiply by T_1, T_2 and T_3 on to each of the other columns which define the other seven left ideals

$$S_2 = S_1 q_{23}, \quad S_3 = S_1 q_{31}, \quad S_4 = S_1 q_{!2}, \quad S_5 = S_1 q_{321}, \quad S_6 = S_1 q_1, \quad S_7 = S_1 q_2, \quad S_8 = S_1 q_3. \tag{3.35}$$

The ideals as written just here are not entirely accurate because the way it is written gives the impression that each of the left ideals S_i have the same a_j coefficients that S_1 does, but in general this need not be the case. We see that across all eight left ideals, the third and fourth row elements transform as one component of a weak isospin doublet with the elements of the fifth and sixth rows transforming as the second component in an isospin doublet. Moreover, we see that across all eight left ideals, the elements from the first, second, seventh and eighth rows transform as isospin singlets. The elements that transform as parts of weak isospin doublets are therefore candidates for left-handed particle states and right-handed antiparticle states whereas the elements that transform as weak isospin singlets are candidates for right-handed particle states and left-handed antiparticle states.

3.5.2 The strong force $SU(3)_c$

Consider the elements

$$\lambda_1 = \frac{i}{2}(\mathfrak{w}_1\tilde{\mathfrak{e}}_2 - \tilde{\mathfrak{e}}_1\mathfrak{e}_2), \quad \lambda_2 = \frac{-i}{2}(\mathfrak{e}_1\mathfrak{e}_2 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_2), \quad \lambda_3 = \frac{i}{2}(\mathfrak{e}_1\tilde{\mathfrak{e}}_1 - \mathfrak{e}_2\tilde{\mathfrak{e}}_2), \tag{3.36}$$

$$\lambda_4 = \frac{i}{2}(\mathfrak{e}_1\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_1\mathfrak{e}_3), \quad \lambda_5 = \frac{-i}{2}(\mathfrak{e}_1\mathfrak{e}_3 + \tilde{\mathfrak{e}}_1\tilde{\mathfrak{e}}_3), \quad \lambda_6 = \frac{i}{2}(\mathfrak{e}_2\tilde{\mathfrak{e}}_3 - \tilde{\mathfrak{e}}_2\mathfrak{e}_3), \tag{3.37}$$

$$\lambda_7 = \frac{-i}{2}(\mathfrak{e}_2\mathfrak{e}_3 + \tilde{\mathfrak{e}}_2\tilde{\mathfrak{e}}_3), \quad \lambda_8 = \frac{i}{\sqrt{3}}(\mathfrak{e}_1\tilde{\mathfrak{e}}_1 + \mathfrak{e}_2\tilde{\mathfrak{e}}_2 - 2\mathfrak{e}_3\tilde{\mathfrak{e}}_3). \tag{3.38}$$

They satisfy the $su(3)$ commutation relations $[\lambda_i, \lambda_j] = if_{ijk}\lambda_k$ where $f_{123} = 1$, $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}$ and $f_{845} = f_{867} = \frac{\sqrt{3}}{2}$ with all other f zero apart from the ones related to the listed nonzero elements by permutation of indices, we are completely antisymmetric. Right multiplying by the λ_i onto the ideals S_i gives:

$$S_1\lambda_i = 0 \qquad \forall\lambda_i \in su(3), \tag{3.39}$$

$$\begin{aligned} S_2\lambda_1 &= S_3, \quad S_2\lambda_2 = iS_3, \quad S_2\lambda_3 = S_2, \quad S_2\lambda_4 = S_4, && (3.40)\\ S_2\lambda_5 &= iS_4, \quad S_2\lambda_6 = 0, \quad S_2\lambda_7 = 0, \quad S_2\lambda_8 = \frac{1}{\sqrt{3}}S_2, && (3.41) \end{aligned}$$

$$\begin{aligned} S_3\lambda_1 &= S_2, \quad S_3\lambda_2 = -iS_2, \quad S_3\lambda_3 = -S_3, \quad S_3\lambda_4 = 0, && (3.42)\\ S_3\lambda_5 &= -iS_2, \quad S_3\lambda_6 = S_4, \quad S_3\lambda_7 = iS_4, \quad S_3\lambda_8 = \frac{1}{\sqrt{3}}S_3, && (3.43) \end{aligned}$$

$$\begin{aligned} S_4\lambda_1 &= 0, \quad S_4\lambda_2 = 0, \quad S_4\lambda_3 = 0, \quad S_4\lambda_4 = S_2, && (3.44)\\ S_4\lambda_5 &= -iS_2, \quad S_4\lambda_6 = S_3, \quad S_4\lambda_7 = -iS_3, \quad S_4\lambda_8 = \frac{-2}{\sqrt{3}}S_4, && (3.45) \end{aligned}$$

$$S_5\lambda_i = 0 \qquad \forall\lambda_i \in su(3), \tag{3.46}$$

$$\begin{aligned} S_6\lambda_1 &= 0, \quad S_6\lambda_2 = -iS_7, \quad S_6\lambda_3 = S_6, \quad S_6\lambda_4 = S_8, && (3.47)\\ S_6\lambda_5 &= -iS_8, \quad S_6\lambda_6 = 0 \quad S_6\lambda_7 = 0, \quad S_6\lambda_8 = \frac{1}{\sqrt{3}}S_6, && (3.48) \end{aligned}$$

$$\begin{aligned} S_7\lambda_1 &= S_6, \quad S_7\lambda_2 = iS_6, \quad S_7\lambda_3 = -S_7, \quad S_7\lambda_4 = 0 && (3.49)\\ S_7\lambda_5 &= 0, \quad S_7\lambda_6 = S_8 \quad S_7\lambda_7 = -iS_8, \quad S_7\lambda_8 = \frac{1}{\sqrt{3}}S_8, && (3.50) \end{aligned}$$

$$\begin{aligned} S_8\lambda_1 &= 0, \quad S_8\lambda_2 = 0, \quad S_8\lambda_3 = 0, \quad S_8\lambda_4 = S_6 && (3.51)\\ S_8\lambda_5 &= iS_6, \quad S_8\lambda_6 = S_7 \quad S_8\lambda_7 = iS_7, \quad S_8\lambda_8 = \frac{-2}{\sqrt{3}}S_8. && (3.52) \end{aligned}$$

Now define $T_\pm \equiv \lambda_1 \pm i\lambda_2$, $V_\pm \equiv \lambda_4 \pm i\lambda_5$, $U_\pm \equiv \lambda_6 \pm i\lambda_7$, $T_3 \equiv \lambda_3$ (not to be confused with the T_3 from the weak force section) and $Y \equiv \frac{2}{\sqrt{3}}\lambda_8$ and recall that given red, yellow and blue color states r, y and b, that

$$\begin{aligned} T_+r &= 0, \quad T_+y = r, \quad T_+ = 0, \quad T_-r = y, \quad T_-y = 0, \quad T_-b = 0, && (3.53)\\ U_+r &= 0, \quad U_+y = 0, \quad U_+b = y, \quad U_-r = 0, \quad U_-y = b, \quad U_-b = 0, && (3.54)\\ V_+r &= 0, \quad V_+y = 0, \quad V_+b = r, \quad V_-r = b, \quad V_-y = 0, \quad V_-b = 0, && (3.55) \end{aligned}$$

$$T_3r = \frac{1}{2}r, \quad T_3y = \frac{-1}{2}y, \quad T_3b = 0, \quad Yr = \frac{1}{3}r, \quad Yy = \frac{1}{3}y, \quad Yb = \frac{-2}{3}b. \tag{3.56}$$

The right action of the λ_i on the left ideal S_3 for example, yields:

$$S_3T_+ = S_2, \quad S_3T_- = 0, \quad S_3U_+ = 0, \quad S_3U_- = S_4, \quad S_3V_+ = 0, \quad S_3V_- = 0, \tag{3.57}$$

$$S_3T_3 = \frac{-1}{2}S_3 \quad \text{and} \quad S_3Y = \frac{1}{3}S_3. \tag{3.58}$$

Comparing this with the known color transformations reveals that $S_2 = S_2^r$, $S_3 = S_3^y$ and $S_4 = S_4^b$, or, in other words, S_2 has red color charge, S_3 has yellow color charge and S_3 has blue color charge and these three ideals S_2, S_3 and S_4 form a color triplet. More precisely, these three left ideals define eight color triplets, one triplet for each row. The left ideal S_1 has eight elements which each transform as a singlet under the 1 representation while the left ideal S_5 has eight elements which each transform as a color singlet under the $\bar{1}$ representation. Moreover one can check that the three left ideals S_6, S_7 and S_8 define eight triplets (one per row) each transforming under the antitriplet $\bar{3}$ representation.

3.5.3 The Lorentz transformations of $\mathbb{C}\ell(6)$

The rotations and boosts of the Lorentz group are generated by elements built from commutation relations of the Dirac γ_μ matrices. The adaption of these matrices to this 8×8 case is as follows. In the chiral representation, otherwise known as the Weyl representation, we have two choices of basis for the 4×4 gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.59}$$

and $(\tilde{\gamma}^0, \tilde{\gamma}^i, \tilde{\gamma}^5) = (\gamma^0, -\gamma^i, -\gamma^5)$. The Dirac representation on the eight-complex-dimensional $\mathbb{C}\ell(6)\mathfrak{p}$ is the direct sum of the two chiral representations:

$$\Gamma^\mu = \begin{pmatrix} \tilde{\gamma}^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \tilde{\gamma}^5 & 0 \\ 0 & \gamma^5 \end{pmatrix}. \tag{3.60}$$

We see that

$$\Gamma^0 S_1 = \begin{pmatrix} a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.61}$$

$$\Gamma^1 S_1 = \begin{pmatrix} -a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.62}$$

$$\Gamma^2 S_1 = \begin{pmatrix} a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.63}$$

$$\Gamma^3 S_1 = \begin{pmatrix} -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.64}$$

and

$$\Gamma^5 S_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.65}$$

The same row interchanges and relative signs between rows hold for the other left ideals under left multiplication by the Γ matrices as well. This means that the first two rows correspond to the two right transforming chiral components of a Weyl spinor for each of the eight left ideals while the third and fourth rows correspond to the two left Weyl components of a Weyl spinor for each of the eight left ideals. Likewise, the fifth and sixth rows correspond to the two left transforming components of a Weyl spinor in each of the eight left ideals while the seventh and eighth rows correspond to the two right transforming components of a Weyl spinor in each of the eight left ideals. Furthermore, the top four rows correspond to a Dirac spinor in each of the eight left ideals while the bottom four rows correspond to another Dirac spinor in each of the eight left ideals.

So far in this presentation we have not looked at the electric charge or the weak hypercharge in detail, but there is already enough data accumulated to know that the sum of the eight left ideals must look as follows if we are going to successfully be able to match standard model fermionic particle states with the left ideals of $\mathbb{C}\ell(6)$:

$$\sum_{i=1}^{8} S_i = \begin{pmatrix} \nu_{R_1} & u^r_{R_1} & u^y_{R_1} & u^b_{R_1} & \bar{e}_{L1} & \bar{d}^{\bar{r}}_{L1} & \bar{d}^{\bar{y}}_{L1} & \bar{d}^{\bar{b}}_{L1} \\ \nu_{R_2} & u^r_{R_2} & u^y_{R_2} & u^b_{R_2} & \bar{e}_{L2} & \bar{d}^{\bar{r}}_{L2} & \bar{d}^{\bar{y}}_{L2} & \bar{d}^{\bar{b}}_{L2} \\ \nu_{L_1} & u^r_{R_1} & u^y_{L_1} & u^b_{L_1} & \bar{e}_{R1} & \bar{d}^{\bar{r}}_{R1} & \bar{d}^{\bar{y}}_{R1} & \bar{d}^{\bar{b}}_{R1} \\ \nu_{L_2} & u^r_{R_2} & u^y_{L_2} & u^b_{L_2} & \bar{e}_{R2} & \bar{d}^{\bar{r}}_{R2} & \bar{d}^{\bar{y}}_{R2} & \bar{d}^{\bar{b}}_{R2} \\ e_{L1} & d^r_{L1} & d^y_{L1} & d^b_{L1} & \bar{\nu}_{R1} & \bar{u}^{\bar{r}}_{R1} & \bar{u}^{\bar{y}}_{R1} & \bar{u}^{\bar{b}}_{R1} \\ e_{L2} & d^r_{L2} & d^y_{L2} & d^b_{L2} & \bar{\nu}_{R2} & \bar{u}^{\bar{r}}_{R2} & \bar{u}^{\bar{y}}_{R2} & \bar{u}^{\bar{b}}_{R2} \\ e_{R1} & d^r_{R1} & d^y_{R1} & d^b_{R1} & \bar{\nu}_{L1} & \bar{u}^{\bar{r}}_{L1} & \bar{u}^{\bar{y}}_{L1} & \bar{u}^{\bar{b}}_{L1} \\ e_{R2} & d^r_{R2} & d^y_{R2} & d^b_{R2} & \bar{\nu}_{L2} & \bar{u}^{\bar{r}}_{L2} & \bar{u}^{\bar{y}}_{L2} & \bar{u}^{\bar{b}}_{L2} \end{pmatrix}. \tag{3.66}$$

3.5.4 Deriving the maximal totally isotropic electro-color symmetry group $U(3)_{ec}$

Take $\alpha \equiv b_1 q_1 + b_2 q_2 + b_3 q_3$ and $\beta \equiv c_1 q_1 + c_2 q_2 + c_3 q_3$ where $b_i, c_i \in \mathbb{C}$. Now for a hermitian operator:

$$\mathcal{H} = \beta^\dagger \alpha + \alpha^\dagger \beta. \tag{3.67}$$

Expanding out all the terms gives $r_0 Q + \sum_{i=1}^{8} r_i \lambda_i$ where $r_0, r_i \in \mathbb{R}$ and where the λ_i above, satisfying the commutation relations of $su(3)$ while the electric charge generator explicitly is $Q = \mathfrak{e}_1 \tilde{\mathfrak{e}}_1 + \mathfrak{e}_2 \tilde{\mathfrak{e}}_2 + \mathfrak{e}_3 \tilde{\mathfrak{e}}_3 \in \mathrm{End}_{\mathbb{C}}(\chi)$.

3.5.5 Electric charge

Given $\bigwedge^\circ \chi$ and $\bigwedge^\circ \chi^\dagger$ the $U(1)_{em}$ action behaves like the exterior product. The electric charge is proportional to the degree with $\frac{k}{3}e$ where $k = 0, \pm 1, \pm 2, \pm 3$ with the identification $\bigwedge^{-k} \chi = \bigwedge^k \chi^\dagger = \bigwedge^k \bar{\chi}$.

We have completed the full standard model identification of one full generation of standard model elementary fermion states. Before looking at other things such as gravity, mass, the Dirac equation and the origins of the electroweak CKM and $PMNS$ mixing matrices, we first have a section devoted to three generations of fermions. But before that we review some foundational parts of Electroweak theory.

3.6 Brief overview of some important parts of the Electroweak Theory

The point of this section is to provide the minimum necessary background within which to read Sec. (3.6.3). In this section we introduce some of the main components of the Electroweak Theory. We mention such concepts as multiplets and symmetry currents.

In 1956, Lee and Yang predicted that parity is violated in weak interactions [51]. In 1957, Wu observed that in β decay, there was a dependence of the angular distribution of decaying electrons on the polarization of the decaying nucleus [74]. The observed decay angular distribution contained both scalar and pseudoscalar quantities. The current-current interactions describing weak processes were modified to incorporate the resulting parity violation by including axial vector terms like $\bar{u}\gamma^\mu\gamma_5 u$ to the currents which already had polar vector terms like $\bar{u}\gamma^\mu u$. Hence, weak currents took on the well known 'V-A' structure. This 'V-A' structure required the conclusion that the weak field quanta must be *vector* particles. The short range of the weak force led to the conclusion that these vector particles must have mass.

That these massive vector particles should be part of a gauge theory, was corroborated by experiments showing that the weak charge g proved to be a universal coupling strength associated in all weak interactions involving both leptons and quarks. Another argument in favor of incorporating those vector particles into a gauge theory was the demand of renormalizability. Gauge theories are in general renormalizable. The non-zero mass of the vector particles seemed to imply that they could not constitute the gauge field quanta in a gauge theory, by virtue of breaking explicit gauge invariance, but this problem was overcome by incorporating the Higgs mechanism [26][27][19] which has the concept of a *vacuum screening current* [2, ch.13] which interacts with the weak gauge field quanta in such a way that the weak gauge field quanta behave as if they have mass. The vacuum screening currents are said to generate the mass of the weak gauge quanta. The field that the weak particles acquire their mass by interacting with, is the Higgs field. The quanta of the Higgs field is the Higgs boson. In 1971, 't Hooft [28] showed that these sorts of theories where massive vector particles acquire mass through vacuum screening currents, are renormalizable.

Another feature related to weak gauge interactions is that there are weak gauge quanta, the $W^\pm$ particles, which have electromagnetic interactions. Thus, this whole vacuum screening process had to be understood in order to properly understand relevant electromagnetic phenomena too. The Standard Model treats electromagnetic currents as being closely connected to the neutral weak currents corresponding to the neutral massive vector Z^0 particle.

Another important feature of the Electroweak Theory is that only left-handed components of weak quark currents participate in weak interactions, and similarly with weak interactions for the generational lepton pairs (ν_e, e^-), (ν_μ, μ^-), (ν_τ, τ^-). The gauge group $SU(2)_L \times U(1)$ describing this Electroweak Theory was first proposed by Glashow (1961). Weinberg and Salam treated the symmetry group as a hidden one (hidden in the sense of spontaneous symmetry breaking [65, p.290][3, p.193][70, p.163]). The resulting Electroweak Theory is in agreement with all known electroweak phenomena.

3.6.1 Standard Model doublets

The (hidden, or spontaneously broken) $SU(2)_L \times U(1)$ *local* gauge transformations mediate interactions between members of an associated symmetry multiplet. The multiplets of the Electroweak Theory are doublets respecting the *weak isospin* symmetry [65, sec.8.5], so the $t = \frac{1}{2}$ representation of $SU(2)_L$ consisting of 2×2 matrices $\frac{1}{2}\tau_i$ are used, where the τ_i are numerically identical to the Pauli matrices σ_i. The doublets have as their elements, Dirac fields. The quark doublets are of the form

$$q_L(u, d_c) = \begin{pmatrix} u \\ d_c \end{pmatrix}_L \quad q_L(c, s_c) = \begin{pmatrix} c \\ s_c \end{pmatrix}_L, \quad q_L(b, t_c) = \begin{pmatrix} t \\ b_c \end{pmatrix}_L. \tag{3.68}$$

where d_c, s_c and b_c are related to the Dirac quark matter fields d, s and b by

$$\begin{pmatrix} d_c \\ s_c \\ b_c \end{pmatrix} = U_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{3.69}$$

U_{CKM} is the Cabibbo-Kobayashi-Maskawa mixing matrix. The unitary Cabibbo-Kobayashi-Maskawa mixing matrix U_{CKM} contains three mixing angles and one CP-violating phase. The numerical values for these angles can be found in, for example, [4]. This flavor mixing was introduced into the Electroweak Theory because (u, c, t) quarks can be changed into any (d, s, b) quarks and vice versa by the absorbing or emitting of a W boson. Charge conservation is conserved because this is consistent with the charges of the top quark entries in the SM doublets differing from the charges of the bottom quark entries in the SM doublets by one unit of charge. The (u, c, t) quarks all have electric charge $\frac{2}{3}$ where as the (d, s, b) quarks all have electric charge $\frac{-1}{3}$. The square of each entry in the matrix U_{CKM} gives the probabilities for various weakly induced quark flavor transformations to occur. The probabilities show that each quark has a strong tendency to change into the flavor that is contained in the same doublet. Quark flavor changing interactions are only known to be brought about by electroweak interactions.

The lepton Standard Model doublets are grouped together as follows:

$$l_e = \begin{pmatrix} \nu_{e,m} \\ e^- \end{pmatrix}_L, \quad l_\mu = \begin{pmatrix} \nu_{\mu,m} \\ \mu^- \end{pmatrix}_L, \quad l_\tau = \begin{pmatrix} \nu_{\tau,m} \\ \tau^- \end{pmatrix}_L \tag{3.70}$$

where

$$\begin{pmatrix} \nu_{e,m} \\ \nu_{\mu,m} \\ \nu_{\tau,m} \end{pmatrix} = U_{\text{MNS}} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{3.71}$$

with U_{MNS} being the Pontecorvo-Maki-Nakagawa-Sakatu (MNS) matrix [58], the leptonic analogue of the CKM matrix U_{CKM} introduced into the Standard Model because of the observation that neutrinos undergo flavor oscillations over macroscopic distances.

The pairs of fields in these quark and lepton doublets are not mass degenerate but this is fine because the symmetry is hidden.

Because the weak force only interacts through left-handed fermion fields, the left-handed fermion fields which have non-zero isospin, are grouped in doublets whereas the right-handed fermion fields form singlets. As a result, there are two covariant derivatives. The left-handed covariant derivative, D_L^μ, for spin-$\frac{1}{2}$ particles is

$$D_L^\mu = \partial^\mu + ig\frac{1}{2}\boldsymbol{\tau} \cdot \mathbf{W}^\mu - ig'\frac{1}{2}B^\mu, \tag{3.72}$$

while the right-handed covariant derivative, D_R^μ, is

$$D_R^\mu = \partial^\mu - ig'B^\mu. \tag{3.73}$$

D_L^μ acts on left-handed leptons which have isospin-$\frac{1}{2}$ and hypercharge $y = -1$ corresponding to the new global $U(1)$ symmetry. The covariant derivative D_R^μ acts on singlet right-handed leptons that have an isospin of zero, and a hypercharge value of $y = -2$. There are two fundamental constants above, g and g', called the weak charges. The weak charge g is from the $SU(2)_L$ part of the gauge group while g' is the weak charge corresponding to the $U(1)$ part of the gauge group. We here speak of hyper charge rather than charge because the field B^μ is not the Standard Model photon field A^μ, but rather related to it in a way to be explained soon. First however, we introduce the Higgs field ϕ, and then explain how to get the vacuum screening currents responsible for giving mass to the other-wise massless $W^\pm$ and Z^0 gauge bosons.

3.6.2 The Higgs field, vacuum screening currents, and $W^{\pm}$, Z^0 and A^{μ} Gauge Fields

The screening currents which provide terms that give mass to the gauge quanta of the weak force, come about by having an absolute vacuum away from where the vacuum expectation value of the scalar field ϕ is zero. The Higgs field has a non-zero vacuum expectation value at any of an infinite number of vacua that are all related to each other by a $U(1)$ transformation. The picking of a *particular* vacuum breaks the symmetry. We will not try to motivate the form of the Higgs doublet used here, but instead refer the reader to the literature, and carry on highlighting some core parts of the Electroweak Theory.

The Higgs field doublet

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}[f + \rho(x)] \end{pmatrix}, \tag{3.74}$$

with f a constant and $\rho(x)$ a scalar field with vanishing vacuum expectation value, is a spinless scalar field with a Lagrangian of the form

$$\mathcal{L}(\phi) = (D_\mu\phi)^\dagger(D^\mu\phi) + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{1}{2}\frac{\mu^2}{f^2}(\phi^\dagger\phi)^2 + \text{interaction terms} \tag{3.75}$$

where the covariant derivatives are the ones such that $\mathcal{L}(\phi)$ is invariant under the hidden gauge group $SU(2)_L \times U(1)$. The *interaction terms* mentioned here are referring to the Yukawa couplings between the Higgs field and the otherwise massless fermionic fields [65, p.310][2, p.465]. The $\frac{f}{\sqrt{2}}$ is the value of the non-zero vacuum expectation value $\langle 0|\,\phi\,|0\rangle$ of the Higgs field.

To find the vacuum screening currents, we consider the terms in this Lagrangian that come from the covariant derivatives and we ignore all of the interaction terms that arise from the ρ part of the Higgs field and just consider that part of the Higgs field concerning the pure vacuum with vacuum value $\frac{f}{\sqrt{2}}$. That is, we consider:

$$j^{a\mu}W_{a\mu} + j^{\mu}B_{\mu} \tag{3.76}$$

where

$$j^{a\mu} = \frac{ig}{2}\left[\phi^\dagger\boldsymbol{\tau}^a(\partial^\mu\phi) - (\partial^\mu\phi)^\dagger\boldsymbol{\tau}^a\phi\right] - \frac{g^2}{2}\phi^\dagger\phi W^{a\mu} - \frac{gg'}{2}\phi^\dagger\boldsymbol{\tau}^a\phi B^\mu \tag{3.77}$$

and

$$j^{\mu} = \frac{ig'}{2}\left[\phi^\dagger(\partial^\mu\phi) - (\partial^\mu\phi)^\dagger\phi\right] - \frac{gg'}{2}\phi^\dagger\boldsymbol{\tau}\phi\cdot\mathbf{W}^\mu - \frac{1}{2}g'^2\phi^\dagger\phi B^\mu. \tag{3.78}$$

Given that we are ignoring the terms involving $\rho(x)$ in the Higgs doublet, it follows that for $a = 1, 2$:

$$j^{1\mu} = \frac{-g^2f^2}{4}W^{1\mu} = -M_W^2W^{1\mu} \quad \text{and} \quad j^{2\mu} = \frac{-g^2f^2}{4}W^{2\mu} = -M_W^2W^{2\mu} \tag{3.79}$$

where

$$M_W \equiv \frac{gf}{2}. \tag{3.80}$$

With $M' \equiv \frac{g'f}{2}$, the $a = 3$ current $j^{3\mu}$ and the current j^μ take the form:

$$j^{3\mu} = -M_W^2W^{3\mu} + MM'B^\mu \quad \text{and} \quad j^\mu = M_WM'W^{3\mu} - M'^2B^\mu. \tag{3.81}$$

These currents mix up the $W^{3\mu}$ and B^μ fields, each of which have indefinite mass leading to the conclusion that these are not the physical fields. If we define

$$g = \sqrt{g^2 + g'^2}\cos(\theta_W), \quad g' = \sqrt{g^2 + g'^2}\sin(\theta_W) \tag{3.82}$$

where θ_W is the Weinberg angle [65, p.312], and also define two fields Z^μ and A^μ by the linear combinations

$$W^{3\mu} = \cos(\theta_W)Z^\mu + \sin(\theta_W)A^\mu \quad \text{and} \quad B^\mu = -\sin(\theta_W)Z^\mu + \cos(\theta_W)A^\mu, \tag{3.83}$$

then a direct calculation reveals that $j^{3\mu}W_{3\mu} + j^\mu B_\mu$ becomes

$$-M_Z^2 Z^\mu Z_\mu \tag{3.84}$$

where $M_Z \equiv \frac{f}{2}\sqrt{g^2 + g'^2} = \frac{M_W}{\cos(\theta_W)}$. Thus, ignoring the self interaction terms arising from the $\rho(x)$ part of the Higgs field that would create a self interaction current $j^\mu(W)$ on the right hand side, the dynamical equations of motion for the gauge fields $W^{1\mu}, W^{2\mu}, Z^\mu$ and A^μ are

$$(\partial_\nu\partial^\nu + M_W^2)W^{1\mu} - \partial^\mu\partial_\nu W^{1\nu} = 0, \quad (\partial_\nu\partial^\nu + M_W^2)W^{2\mu} - \partial^\mu\partial_\nu W^{2\nu} = 0 \tag{3.85}$$

and

$$(\partial_\nu\partial^\nu + M_Z^2)Z^\mu - \partial^\mu\partial_\nu Z^\nu = 0, \quad \partial_\nu\partial^\nu A^\mu - \partial^\mu\partial_\nu A^\nu = 0. \tag{3.86}$$

By rewriting the left-handed covariant derivative D_L^μ in terms of the fields Z^μ and A^μ instead of $W^{3\mu}$ and B^μ, we can identify A^μ with the photon field by the relation

$$e = g\sin(\theta_W) \tag{3.87}$$

where e is the unit of electromagnetic charge.

3.6.3 The electroweak bosons and the Weinberg angle

We have seen in this paradigm that the electric charge and weak isospin are more fundamental than the weak hypercharge. Also, the electromagnetic and weak interactions are seen to be more fundamental than the electroweak interaction. The usual electroweak symmetry breaking apparently is emergent from the structure we have here described. In this section we follow [68] in explaining how there is an inherent prediction of the electroweak Weinberg angle which exists to be elucidated. This is done here.

We need to first define a hermitian inner product on ideals. Firstly, recall that a basis of the ideal $\bigwedge^\circ \chi^\dagger\mathfrak{p}$ can be given by

$$(1\mathfrak{p}, q_{23}^\dagger\mathfrak{p}, q_{31}^\dagger\mathfrak{p}, q_{12}^\dagger\mathfrak{p}, q_{321}^\dagger\mathfrak{p}, q_1^\dagger\mathfrak{p}, q_2^\dagger\mathfrak{p}, q_3^\dagger\mathfrak{p}). \tag{3.88}$$

The hermitian inner product is given by

$$\mathfrak{h}(a,b)\mathfrak{q}^\dagger\mathfrak{q} = (a^\dagger\mathfrak{q})^\dagger b^\dagger\mathfrak{q} = \mathfrak{q}^\dagger ab^\dagger\mathfrak{q}, \quad \forall a,b \in \overset{\circ}{\bigwedge}\chi. \tag{3.89}$$

For the proof, see Appendix A of [68]. Consider an electroweak doublet $(\beth, \gimel)^t$. Setting notation for subspaces, we have

$$\beth \in \mathbb{W}_0, \quad \gimel \in \mathbb{W}_1, \tag{3.90}$$

which, when acted upon by the projection operators $P_{L/R} \equiv \frac{1}{2}(1 \mp \gamma^5)$ allows the definition of four subspaces:

$$\mathbb{W}_{0L/R} \equiv P_{L/R}\mathbb{W}_0, \quad \mathbb{W}_{1L/R} \equiv P_{L/R}\mathbb{W}_1. \tag{3.91}$$

The generators T_{jL} of the weak symmetry group $SU(2)_L$ are defined by

$$T_{jL} \in \mathrm{End}_{\mathbb{C}}(\mathbb{W}_{0L} \oplus \mathbb{W}_{1L}), \quad T_{jL} \equiv \sigma_j \otimes 1_2. \tag{3.92}$$

To find a representation of $SU(2)_L$ taking into account the chirality of each space, we start with the elements

$$\omega_u = \begin{pmatrix} 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega_d = \begin{pmatrix} 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \omega_0 = \begin{pmatrix} \sigma_+ & 0 & 0 & 0 \\ 0 & -\sigma_+ & 0 & 0 \\ 0 & 0 & -\sigma_+ & 0 \\ 0 & 0 & 0 & \sigma_+ \end{pmatrix} \tag{3.93}$$

Now define the complex null vector spaces ק and $\text{ק}^\dagger$ by

$$\text{ק} \equiv \mathrm{span}_{\mathbb{C}}(\omega_u, \omega_d, \omega_o), \quad \text{ק}^\dagger \equiv \mathrm{span}_{\mathbb{C}}(\omega_u^\dagger, \omega_d^\dagger, \omega_o^\dagger). \tag{3.94}$$

It turns out that the ω_j's together with their adjoints form a Witt basis for $\text{ק}^\dagger \oplus \text{ק}$ so that

$$\{\omega_j, \omega_k\} = 0 = \{\omega_j^\dagger, \omega_k^\dagger\}, \quad \{\omega_j, \omega_k^\dagger\} = \delta_{jk}. \tag{3.95}$$

If we define $\omega \equiv \omega_u \omega_d \omega_o$ and $\omega^\dagger \equiv \omega_o^\dagger \omega_d^\dagger \omega_u^\dagger$ then ω and $\omega^\dagger$ are nilpotents and moreover, the resulting idempotents are $\omega\omega^\dagger = \mathfrak{p}$ and $\omega^\dagger\omega = \mathfrak{p}^\dagger$. One more thing we need is to define ($\mathbb{W}_\omega \equiv \mathrm{span}_{\mathbb{C}}(\omega_u^\dagger, \omega_d^\dagger), h_W$) where h_W is the restriction of $h_{\text{ק}}$ to $\mathbb{W}_\omega$. Next, we take note that $\mathbb{W}_{0R} = 1\mathrm{span}_{\mathbb{C}}(\mathfrak{p}, \omega_0^\dagger \mathfrak{p})$, $\mathbb{W}_{0L} = \omega_u^\dagger \mathbb{W}_{0R}$, $\mathbb{W}_{1R} = \omega_u^\dagger \omega_d^\dagger \mathbb{W}_{0R}$ and $\mathbb{W}_{1L} = \omega_d^\dagger \mathbb{W}_{0R}$.

Mathematically, the electroweak bosons are connections in the gauge bundle, the fiber of which is the two-dimensional (hermitian) vector space $(\mathbb{W}_\omega, h_W)$ with $\mathbb{W}_\omega \equiv \mathrm{span}_{\mathbb{C}(\omega_u^\dagger, \omega_d^\dagger)}$. The electroweak bosons internal components are elements of $u(2)_{\mathrm{ew}} \cong u(\mathbb{W}_\omega)$. As a vector space, this is four-real-dimensional. A decomposition can be written as

$$u(\mathbb{W}_\omega) = \gamma(\mathbb{W}_\omega) \oplus W(\mathbb{W}_\omega) \oplus Z(\mathbb{W}_\omega) \tag{3.96}$$

with obvious notation implying that for the photons γ, the $W^\pm$ bosons and the Z^0 boson we have respectively $\gamma \in \gamma(\mathbb{W}_\omega)$, $W^\pm \in W(\mathbb{W}_\omega)$ and $Z^0 \in Z(\mathbb{W}_\omega)$. At first sight it would appear that there are infinitely many ways to form this decomposition. However, the exact decomposition corresponding to reality is uniquely determined by both the Higgs field and the Weinberg mixing angle θ_W. The Higgs field ϕ together with an inner product on $u(\mathbb{W}_\omega)$ picks out a special complex line in $\mathbb{W}_\omega$. The inner product, on account of Ad-invariance, takes the form (with $r_2 > 0$ a constant)

$$\langle a, b\rangle_{u(\mathbb{W}_\omega)} = -2r_2 g'^2 \mathrm{Tr}(ab) + r_2(g'^2 - g^2)\mathrm{Tr}(a)\mathrm{Tr}(b), \quad a, b \in u(\mathbb{W}_\omega). \tag{3.97}$$

The spacetime scalar Higgs field internally is a vector $\phi \in \mathbb{W}_\omega$ with direction given by

$$\omega_u^\dagger = \frac{\phi}{\sqrt{h_W(\phi,\phi)}}. \tag{3.98}$$

Mathematically the Higgs field is a section of a one-dimensional complex subbundle of the electroweak bundle. The projection operators which project onto this Higgs subbundle, or onto the orthogonal subbundle are $\frac{1}{2}(1 \mp i\mathfrak{e}\tilde{\mathfrak{e}})$.

To derive the prediction for the Weinberg angle, note firs that for $su(N)$ we have a unique Ad-invariant inner product given by

$$\langle A, B\rangle_{SU(N)} = -Nr_N \mathrm{Tr}(AB), \quad A, B \in su(N), \quad r_N > 0. \tag{3.99}$$

For any $A \in su(N)$, $\mathrm{Tr}(A) = 0$ so the embedding of $u(2)$ in $su(N)$ must be traceless. Thus for any $a \in u(2)$ a basis in $\mathbb{C}^N$ that contains a basis in $\mathbb{W}_\omega$ is given by

$$a \mapsto a \oplus \left(-\frac{1}{N-2}\mathrm{Tr}(aI_{W_\omega^\perp})\right). \tag{3.100}$$

It then follows that

$$\begin{aligned}
\langle a, b\rangle_{u(2)} &= \left\langle a \oplus \left(-\frac{1}{N-2}\mathrm{Tr}(aI_{\mathbb{W}_\omega^\perp})\right), b \oplus \left(-\frac{1}{N-2}\mathrm{Tr}(bI_{\mathbb{W}_\omega^\perp})\right)\right\rangle_{SU(N)} & (3.101)\\
&= -Nr_N\mathrm{Tr}(ab) - Nr_N\left(-\frac{1}{N-2}\right)^2 \mathrm{Tr}(a)\mathrm{Tr}(b)\mathrm{Tr}(I_{\mathbb{W}_\omega^\perp}) & (3.102)\\
&= -Nr_N\mathrm{Tr}(ab) - Nr_N\frac{1}{N-2}\mathrm{Tr}(a)\mathrm{Tr}(b). & (3.103)
\end{aligned}$$

This has to be equal to

$$\langle a, b\rangle_{u(\mathbb{W}_\omega)} = -2r_2 g^{'2}\mathrm{Tr}(ab) + r_2(g^{'2} - g^2)\mathrm{Tr}(a)\mathrm{Tr}(b), \quad a, b \in u(\mathbb{W}_\omega) \tag{3.104}$$

from which it follows that $2r_2g^{'2} = r_N N$ and $r_2(g^{'2} - g^2) = -Nr_N\frac{1}{N-2}$. It then follows that

$$\sin^2(\theta_{W,N}) = \frac{g^{'2}}{g^2 + g^{'2}} = \frac{\frac{N}{2}}{N + \frac{N}{N-2}} = \frac{1}{2}\frac{N-2}{N-1}. \tag{3.105}$$

Since $u(\mathbb{W}_\omega)$ is embedded in $su(3)$ and $su(3)$ is the symmetry group of $(ק^\dagger, h_ק)$ we set $N = 3$ to obtain

$$\sin^2(\theta_{W,\mathbb{C}\ell(6)}) = \sin^2(\theta_{W,3}) = \frac{1}{4} = 0.25, \quad \to \quad \theta_{W,\mathbb{C}\ell(6)} = \frac{\pi}{6}\,\mathrm{rad}. \tag{3.106}$$

The experimental data for the Weinberg angle is

1. $\sin^2\theta_W \approx 0.223 - 0.24$ [20].
2. $\sin^2\theta_W \approx 0.23129$ [60].

At this point we point out that so far we have only been looking at one copy of $\mathbb{J}_3^{\mathbb{O}}$. The experimental values were taken in the real world where there exists more than one particle, and with different co-adjacency relations between them (i.e. different quantum spacetime data relative to each other). Given that our fundamental premise is that the universe is one inseparable entangled quantum state which self-interacts, there is no such notion as taking a particle in isolation and describing it without regard for anything else that it is inseparably linked with. When we progress from one particle to more than one particle and look at quantum spacetime, we will see that it does not suffice to take commuting copies of $J_3^{\mathbb{O}}$. In other words we should not use the tensor product to bootstrap more states into the universe. As will be discussed later, what we naturally need is the braided tensor product

$$J_3^{\mathbb{O}} \underline{\otimes} J_3^{\mathbb{O}} \underline{\otimes} J_3^{\mathbb{O}} \underline{\otimes} \cdots \underline{\otimes} J_3^{\mathbb{O}} \tag{3.107}$$

which deforms the entire mathematical category of objects. Hence a more complete calculation will take into account q-deforming. We thus consider the non-q-deformed prediction for the Weinberg angle as being encouraging since the undeformed value is close, as it should be, to the experimental value, but not completely on the mark, also in keeping with expectation, since we have not yet q-deformed.

3.7 Three full generations of elementary fermions from the Sedenions $\mathbb{S} \equiv \mathbb{A}_4$

In this section we follow [66] and [14]. The sedenions $\mathbb{S}$ are the Cayley-Dickson algebra $\mathbb{A}_4$ with four generators, say i, j, k, l. Let us take the complex numbers $\mathbb{C}$ as the Cayley-Dickson algebra generated by i, the quaternions $\mathbb{A}_2$ generated by i, j, and the octonions $\mathbb{A}_3$ generated by i, j, k. Cayley-Dickson algebras have various $*$-representations which basically just correspond to different ways of grouping the terms together. The sedenions have four generators i, j, k, l and can be written as

$$s = \sum_{i=0}^{15} s_i \mathbf{e}_i = \sum_{j=0}^{2^{4-m}-1} \beta_j(s) \mathbf{e}_{2^m j}. \tag{3.108}$$

With $\beta_j(s) \in A_m$ and $\mathbf{e}_0 \equiv 1$ we have the representations denoted by $\mathbb{S} = A_m * B(m, 4)$ as follows:

$$\text{Case 1} \quad m = 4: \quad s = \beta_0(s)1 \quad \in \mathbb{S} * \mathbb{R}, \;\; \beta_0(s) \in \mathbb{S} \tag{3.109}$$

$$\text{Case 2} \quad m = 3: \quad s = \beta_0(s)1 + \beta_1(s)l \quad \in \mathbb{O} * \mathbb{C}_l, \;\; \beta_j(s) \in \mathbb{O}, \tag{3.110}$$

$$\text{Case 3} \quad m = 2: \quad s = \beta_0(s)1 + \beta_1(s)k + \beta_2(s)l + \beta_3(s)kl \quad \in \mathbb{H} * \mathbb{H}_{k,l} \tag{3.111}$$

$$\beta_m(s) = r_{0m}1 + r_{1m}i + r_{2m}j + r_{3m}ij$$

$$\text{Case 4} \quad m = 1 \;\; s = c_0 + c_1 j + c_2 k + c_3 l + c_4 kl + c_5 jl + c_6 jk + c_7 jkl \quad \in \mathbb{C} * \mathbb{O}_{j,k,l}, \tag{3.112}$$

$$\beta_m(s) = c_{m0}1 + c_{m1}i$$

$$\text{Case 5} \quad m = 0: \quad s = \sum_{n=0}^{15} \beta_n(s)\mathbf{e}_n, \;\; \in \mathbb{R} * \mathbb{S}_{i,j,k,l}, \;\; \beta_n \in \mathbb{R}. \tag{3.113}$$

The multiplication laws are inherited by the multiplication table in $\mathbb{S}$ with $\mathbf{e}_{2^m j + l} = \mathbf{e}_l \mathbf{e}_{2^m j}$. By using the sedenion multiplication table in the appendices, we can write a sedenion explicitly in the various ways as follows:

$$\begin{aligned}
\mathbb{S} \ni s &= s_0 1 + s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3 + s_4\mathbf{e}_4 + s_5\mathbf{e}_5 + s_6\mathbf{e}_6 + s_7\mathbf{e}_7 \\
&+ s_8\mathbf{e}_8 + s_9\mathbf{e}_9 + s_{10}\mathbf{e}_{10} + s_{11}\mathbf{e}_{11} + s_{12}\mathbf{e}_{12} + s_{13}\mathbf{e}_{13} + s_{14}\mathbf{e}_{14} + s_{15}\mathbf{e}_{15} \\
&= (s_0 + s_1\mathbf{e}_1 + s_2\mathbf{e}_3 + s_4\mathbf{e}_4 + s_5\mathbf{e}_5 + s_6\mathbf{e}_6 + s_7\mathbf{e}_7) \\
&+ (s_8 + s_9\mathbf{e}_1 + s_{10}\mathbf{e}_2 + s_{11}\mathbf{e}_3 + s_{12}\mathbf{e}_4 + s_{13}\mathbf{e}_5 + s_{14}\mathbf{e}_6 + s_{15}\mathbf{e}_7)\mathbf{e}_8 \in \mathbb{O} * \mathbb{C}_{\mathbf{e}_8} \\
&= (s_0 + s_4\mathbf{e}_4 + s_9\mathbf{e}_9 + s_{13}\mathbf{e}_{13})1 + (s_1 - s_5\mathbf{e}_4 + s_8\mathbf{e}_9 - s_{12}\mathbf{e}_{13})\mathbf{e}_1 \\
&+ (s_2 - s_6\mathbf{e}_4 - s_{11}\mathbf{e}_9 + s_{15}\mathbf{e}_{13})\mathbf{e}_2 + (s_3 - s_7\mathbf{e}_4 + s_{10}\mathbf{e}_9 - s_{14}\mathbf{e}_{13})\mathbf{e}_3 \in \mathbb{H} * \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2} \\
&= (s_0 + s_8\mathbf{e}_8) + (s_1 - s_9\mathbf{e}_8)\mathbf{e}_1 + (s_2 - s_{10}\mathbf{e}_8)\mathbf{e}_2 + (s_3 - s_{11}\mathbf{e}_8)\mathbf{e}_3 \\
&+ (s_4 - s_{12}\mathbf{e}_8)\mathbf{e}_4 + (s_5 - s_{13}\mathbf{e}_8)\mathbf{e}_5 + (s_6 - s_{14}\mathbf{e}_8)\mathbf{e}_6 + (s_7 - s_{15}\mathbf{e}_8)\mathbf{e}_7 \in \mathbb{C} * \mathbb{O}_{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_7}.
\end{aligned}$$

The $\mathbb{O} * \mathbb{C}_{\mathbf{e}_8}$ is the usual Cayley-Dickson representation of a sedenion. It turns out that these representations are not all equivalent in the sense that not all of these representations are invariant under the full $\mathrm{Aut}(\mathbb{S}) = G_2 \times S_3$.[1] While $\mathbb{S} * \mathbb{R}$ and $\mathbb{O} * \mathbb{C}_{\mathbf{e}_8}$ are invariant under all of $\mathrm{Aut}(\mathbb{S})$, the three representations $\mathbb{H} * \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$, $\mathbb{C} * \mathbb{O}_{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_7}$ and $\mathbb{R} * \mathbb{S}_{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_7,\mathbf{e}_8}$ are not invariant under all of $\mathrm{Aut}(\mathbb{S})$.

We look now at the eigentheory of $\mathbb{S} = \mathbb{H} * \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$. In general for a nonzero element a of a Cayley-Dickson algebra A_n we have the $\mathbb{R}$-linear map

$$M_a \equiv \frac{1}{|a|^2} L_{a^*} L_a = M_{a^*} \tag{3.114}$$

the eigenvalues of which are the eigenvalues of a and the eigenvectors of which are the eigenvectors of a. We denote by $\mathrm{Eig}_\lambda(a)$ the λ-eigenspace of a. M_a is diagonalizeable with non-negative eigenvalues. Moreover for two distinct eigenvalues the corresponding eigenspaces are orthogonal. Moreover, every nonzero element $a \in A_n$ has at least one eigenvalue being 1 so one always has $\mathrm{Eig}_1(a)$ in addition to any other eigenspaces there may be. Moreover, if we have found all of the eigenvalues, we have $\sum_i \lambda_i = 2^n$. Also, for $\lambda \neq 0$ we have $L_a \in\mathrm{Aut}(\mathrm{Eig}_\lambda(a))$. We can also form the eigendecomposition of $x \in A_n$ in terms of eigenvectors x_i corresponding to distinct eigenvalues λ_i.

Importantly, there exists the following general result:

Let $a \in \mathbb{C}_{n-1}^\perp$, and let $\alpha, \beta \in \mathbb{C}_{n-1}$ such that $|a| = |\alpha|^2 + |\beta|^2 = 1$, so that $(\alpha a, \beta a)$ is a unit vector. Suppose that $\alpha \times \beta \neq 0$ so that α and β are $\mathbb{R}$-linearly independent. Let $\gamma = \frac{\alpha\times\beta}{|\alpha\times\beta|}$. Then we have
(a) $\langle\langle a, i_{n-1}, i_n\rangle\rangle \subset \mathrm{Eig}_1((\alpha a, \beta a))$;
(b)$\{(x, -\gamma x) : x \in \mathrm{Eig}_1(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^\perp\} \subset \mathrm{Eig}_{1+2|\alpha\times\beta|}((\alpha a, \beta a))$;
(c) $\{(x, \gamma x) : x \in \mathrm{Eig}_1(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^\perp\} \subset \mathrm{Eig}_{1-2|\alpha\times\beta|}((\alpha a, \beta a))$;
(d) $\{(x, -\gamma x) : x \in \mathrm{Eig}_\lambda(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^\perp\} \subset \mathrm{Eig}_{1+2|\alpha\times\beta|}((\alpha a, \beta a))$;
(e) $\{(x, \gamma x) : x \in \mathrm{Eig}_\lambda(a) \cap \langle\langle a, i_{n-1}\rangle\rangle^\perp\} \subset \mathrm{Eig}_{1+2|\alpha\times\beta|}((\alpha a, \beta a))$;

[1] $\mathrm{Aut}(\mathbb{O}) = G_2$. Since $SU(3) \subset G_2$ we can see that it is the permutation group S_3 that accommodates the extension from one generation of fermions to three generations of fermions. One might wonder whether we can just keep bootstrapping like this to get more generations. The answer however is that we cannot. Firstly, the automorphism groups do not work out in a conducive way for this for higher Cayley-Dickson algebras. Secondly, as we shall see when we consider $J_3^{\mathbb{O}}$, we are stuck with precisely three generations of elementary fermion states.

Moreover, every eigenvalue of $(\alpha a, \beta a)$ is either equal to 1 or else is of the form $\left(1 \pm \frac{2|\alpha \times \beta|}{|\alpha|^2+|\beta|^2}\right)$ where λ is an eigenvalue of a.

Applying these results to the special case $A_4 = \mathbb{S}$ gives the following. Let a and b be orthogonal, imaginary, nonzero elements of $A_3 = \mathbb{O}$ such that $|a| = |b|$. Then the eigenvalues of $(a, b) \in A_4 = \mathbb{S}$ are 0, 1 and 2 with respective multiplicities of 4, 8 and 4. Moreover,
(a) $\mathrm{Eig}_0((a,b)) = \{(a, \frac{-ab \cdot x}{|ab|}) : x \in \langle\langle a, b\rangle\rangle^{\perp}\}$.
(b) $\mathrm{Eig}_1((a,b)) = \langle\langle a, b\rangle\rangle \times \langle\langle a, b\rangle\rangle$.
(c) $\mathrm{Eig}_2((a,b)) = \{(a, \frac{ab \cdot x}{|ab|}) : x \in \langle\langle a, b\rangle\rangle^{\perp}\}$.

We wish to apply this to the particular representation $A_4 = \mathbb{S} = \mathbb{H} * \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$. In order to get the above results compatible with this particular representation of the sedenions we first define head and tail functions $h_{\mathbf{e}_1}, t_{\mathbf{e}_1} : \mathbb{O} \to \mathbb{O}$ by

$$h_{\mathbf{e}_1}(\alpha) = \frac{1}{2}(\alpha + \mathbf{e}_1\alpha\mathbf{e}_1^*), \quad t_{\mathbf{e}_1}(\alpha) = \frac{1}{2}(\alpha - \mathbf{e}_1\alpha\mathbf{e}_1^*). \tag{3.115}$$

Then any octonions have the decomposition $\mathbb{C}_{\mathbf{e}_1} \oplus \mathcal{D}^3_{\mathbf{e}_1}$. $\mathcal{D}^3_{\mathbf{e}_1}$ is a $\mathbb{C}_{\mathbf{e}_1}$-vector space. The 1-eigenspace of any octonion only depends on its orthogonal projection onto $\mathcal{D}^3_{\mathbf{e}_1}$. One now only need consider $L_{(a,b)}$ where (a, b) is a unit octonion in $\mathcal{D}^3_{\mathbf{e}_1}$ such that a and b are orthogonal pure quaternions with the same norm. We can now use octonionic coefficients in the representation $\mathbb{O} = \mathbb{H} * \mathbb{C}_q$ where $q = \frac{-ab}{|ab|}$. Now define $\alpha, \beta \in \mathbb{C}_q$ such that $|\alpha|^2 + |\beta|^2 = 1$, $|\alpha| = |\beta|$, $\mathbf{g} \in \mathcal{D}^3_q$ and $|\mathbf{g}| = 1$.

There are six multiplication maps $L_{(\alpha\mathbf{g},\beta\mathbf{g})}$ where $(\alpha\mathbf{g}, \beta\mathbf{g})$ are unit sedenions in $\mathcal{D}^4_{\mathbf{e}_1}$. We then get the following **for each of the six** maps $L_{(\alpha\mathbf{g},\beta\mathbf{g})}$:

$$\begin{aligned}
\mathrm{Eig}_0((\alpha\mathbf{g}, \beta\mathbf{g})) &= \{(x, q \cdot x) : x \in \mathbb{O}, h_{\mathbf{e}_1}(x(\beta\mathbf{g})^*) = 0, h_{\mathbf{e}_1}(x(\alpha\mathbf{g})^*) = 0\} \simeq \mathbb{H},\\
\mathrm{Eig}_1((\alpha\mathbf{g}, \beta\mathbf{g})) &= gen\{1, \mathbf{g}, q, \mathbf{e}_1\} \simeq \mathbb{O},\\
\mathrm{Eig}_2((\alpha\mathbf{g}, \beta\mathbf{g})) &= \{(x, -q \cdot x) : x \in \mathbb{O}, h_{\mathbf{e}_1}(x(\beta\mathbf{g})^*) = 0, h_{\mathbf{e}_1}(x(\alpha\mathbf{g})^*) = 0\} \simeq \mathbb{H}.
\end{aligned}$$

Moreover because the multiplication maps are automorphisms we have

$$L_{(\alpha\mathbf{g},\beta\mathbf{g})} : \mathrm{Eig}_0((\alpha\mathbf{g}, \beta\mathbf{g})) \to \mathrm{Eig}_0((\alpha\mathbf{g}, \beta\mathbf{g})) \in SU(2) \tag{3.116}$$
$$L_{(\alpha\mathbf{g},\beta\mathbf{g})} : \mathrm{Eig}_1((\alpha\mathbf{g}, \beta\mathbf{g})) \to \mathrm{Eig}_1((\alpha\mathbf{g}, \beta\mathbf{g})) \in SU(3) \tag{3.117}$$
$$L_{(\alpha\mathbf{g},\beta\mathbf{g})} : \mathrm{Eig}_2((\alpha\mathbf{g}, \beta\mathbf{g})) \to \mathrm{Eig}_2((\alpha\mathbf{g}, \beta\mathbf{g})) \in SU(2) \tag{3.118}$$

We can identify the 0-eigenspace as the $SU(2)_R$ singlets, the 1-eigenspace with the $SU(3)_c$ colour singlets and triplets, the 2-eigenspace as the $SU(2)_L$ doublets, and moreover, there are six multiplication maps implying that there are three copies of these right-handed singlets and left-handed doublets. Hence there are three generations. Finally, the $\mathbb{C}_{\mathbf{e}_1}$ linear conjugation of $L_{(\alpha\mathbf{g},\beta\mathbf{g})}$ gives the corresponding antiparticles. We have also seen when looking at $\mathrm{End}(\mathbb{C} \otimes \mathbb{O})$ that we get two left chiral components and two right chiral components. Putting it all together and we have the following:

0-Eigenspace Singlet States:

$$e^-_{R1},\ \nu_{e,R1},\ \mu^-_{R1},\ \nu_{\mu,R1},\ \tau^-_{R1},\ \nu_{\tau,R1},\ e^+_{L1},\ \bar{\nu}_{e,L1},\ \mu^+_{L1},\ \bar{\nu}_{\mu,L1},\ \bar{\tau}^+_{L1},\ \bar{\nu}_{\tau,L1}$$

$$e^{-}_{R2},\ \nu_{e,R2},\ \mu^{-}_{R2},\ \nu_{\mu,R2},\ \tau^{-}_{R2},\ \nu_{\tau,R2},\ e^{+}_{L2},\ \bar{\nu}_{e,L2},\ \mu^{+}_{L2},\ \bar{\nu}_{\mu,L2},\ \bar{\tau}^{+}_{L2},\ \bar{\nu}_{\tau,L2}$$

$$u^{r}_{R1},\ u^{g}_{R1},\ u^{b}_{R1},\ c^{r}_{R1},\ c^{g}_{R1},\ c^{b}_{R1},\ t^{r}_{R1},\ t^{g}_{R1},\ t^{b}_{R1},\ \bar{u}^{\bar{r}}_{L1},\ \bar{u}^{\bar{g}}_{L1},\ \bar{u}^{\bar{b}}_{L1},\ \bar{c}^{\bar{r}}_{L1},\ \bar{c}^{\bar{g}}_{L1},\ \bar{c}^{\bar{b}}_{L1},\ \bar{t}^{\bar{r}}_{L1},\ \bar{t}^{\bar{g}}_{L1},\ \bar{t}^{\bar{b}}_{L1}$$

$$u^{r}_{R2},\ u^{g}_{R2},\ u^{b}_{R2},\ c^{r}_{R2},\ c^{g}_{R2},\ c^{b}_{R2},\ t^{r}_{R2},\ t^{g}_{R2},\ t^{b}_{R2},\ \bar{u}^{\bar{r}}_{L2},\ \bar{u}^{\bar{g}}_{L2},\ \bar{u}^{\bar{b}}_{L2},\ \bar{c}^{\bar{r}}_{L2},\ \bar{c}^{\bar{g}}_{L2},\ \bar{c}^{\bar{b}}_{L2},\ \bar{t}^{\bar{r}}_{L2},\ \bar{t}^{\bar{g}}_{L2},\ \bar{t}^{\bar{b}}_{L2}$$

2-Eigenspace Doublet States:

$$\begin{pmatrix} \nu_{e,L1} \\ \nu_{e,L2} \\ e^{-}_{L1} \\ e^{-}_{L2} \end{pmatrix},\ \begin{pmatrix} \nu_{\mu,L1} \\ \nu_{\mu,L2} \\ \mu^{-}_{L1} \\ \mu^{-}_{L2} \end{pmatrix},\ \begin{pmatrix} \nu_{\tau,L1} \\ \nu_{\tau,L2} \\ \tau^{-}_{L1} \\ \tau^{-}_{L2} \end{pmatrix},\ \begin{pmatrix} e^{+}_{R1} \\ e^{+}_{R2} \\ \bar{\nu}_{e,R1} \\ \bar{\nu}_{e,R2} \end{pmatrix},\ \begin{pmatrix} \mu^{+}_{R1} \\ \mu^{+}_{R2} \\ \bar{\nu}_{\mu,R1} \\ \bar{\nu}_{\mu,R2} \end{pmatrix},\ \begin{pmatrix} \tau^{+}_{R1} \\ \tau^{+}_{R2} \\ \bar{\nu}_{\tau,R1} \\ \bar{\nu}_{\tau,R2} \end{pmatrix}$$

$$\begin{pmatrix} u^{r}_{L1} \\ u^{r}_{L2} \\ d^{r}_{L1} \\ d^{r}_{L2} \end{pmatrix},\ \begin{pmatrix} u^{g}_{L1} \\ u^{g}_{L2} \\ d^{g}_{L1} \\ d^{g}_{L2} \end{pmatrix},\ \begin{pmatrix} u^{b}_{L1} \\ u^{b}_{L2} \\ d^{b}_{L1} \\ d^{b}_{L2} \end{pmatrix},\ \begin{pmatrix} \bar{d}^{\bar{r}}_{R1} \\ \bar{d}^{\bar{r}}_{R2} \\ \bar{u}^{\bar{r}}_{R1} \\ \bar{u}^{\bar{r}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{d}^{\bar{g}}_{R1} \\ \bar{d}^{\bar{g}}_{R2} \\ \bar{u}^{\bar{g}}_{R1} \\ \bar{u}^{\bar{g}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{d}^{\bar{b}}_{R1} \\ \bar{d}^{\bar{b}}_{R2} \\ \bar{u}^{\bar{b}}_{R1} \\ \bar{u}^{\bar{b}}_{R2} \end{pmatrix} \qquad (3.119)$$

$$\begin{pmatrix} c^{r}_{L1} \\ c^{r}_{L2} \\ s^{r}_{L1} \\ s^{r}_{L2} \end{pmatrix},\ \begin{pmatrix} c^{g}_{L1} \\ c^{g}_{L2} \\ s^{g}_{L1} \\ s^{g}_{L2} \end{pmatrix},\ \begin{pmatrix} c^{b}_{L1} \\ c^{b}_{L2} \\ s^{b}_{L1} \\ s^{b}_{L2} \end{pmatrix},\ \begin{pmatrix} \bar{s}^{\bar{r}}_{R1} \\ \bar{s}^{\bar{r}}_{R2} \\ \bar{c}^{\bar{r}}_{R1} \\ \bar{c}^{\bar{r}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{s}^{\bar{g}}_{R1} \\ \bar{s}^{\bar{g}}_{R2} \\ \bar{c}^{\bar{g}}_{R1} \\ \bar{c}^{\bar{g}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{s}^{\bar{b}}_{R1} \\ \bar{s}^{\bar{b}}_{R2} \\ \bar{c}^{\bar{b}}_{R1} \\ \bar{c}^{\bar{b}}_{R2} \end{pmatrix} \qquad (3.120)$$

$$\begin{pmatrix} t^{r}_{L1} \\ t^{r}_{L2} \\ b^{r}_{L1} \\ b^{r}_{L2} \end{pmatrix},\ \begin{pmatrix} t^{g}_{L1} \\ t^{g}_{L2} \\ b^{g}_{L1} \\ b^{g}_{L2} \end{pmatrix},\ \begin{pmatrix} t^{b}_{L1} \\ t^{b}_{L2} \\ b^{b}_{L1} \\ b^{b}_{L2} \end{pmatrix},\ \begin{pmatrix} \bar{b}^{\bar{r}}_{R1} \\ \bar{b}^{\bar{r}}_{R2} \\ \bar{t}^{\bar{r}}_{R1} \\ \bar{t}^{\bar{r}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{b}^{\bar{g}}_{R1} \\ \bar{b}^{\bar{g}}_{R2} \\ \bar{t}^{\bar{g}}_{R1} \\ \bar{t}^{\bar{g}}_{R2} \end{pmatrix},\ \begin{pmatrix} \bar{b}^{\bar{b}}_{R1} \\ \bar{b}^{\bar{b}}_{R2} \\ \bar{t}^{\bar{b}}_{R1} \\ \bar{t}^{\bar{b}}_{R2} \end{pmatrix}. \qquad (3.121)$$

There are 60 singlet states and 96 doublet states which makes 156 states in total. On top of this there are 8 photon states, 36 weak boson states, making 44 electroweak states and there is also 1 Higgs state. Altogether that makes 201 states. If one includes the gluons, there are 64 gluon states so all together there are 265 particle states.

This completes our examination of the physical content of the quaternions and the octonions. With this background knowledge, we are now ready to move on to considering the exceptional Jordan algebra $J_3^{\mathbb{O}}$. This gives rise to important additional physics such as:

1. Dirac equation.
2. The origin of the electroweak CKM and $PMNS$ mixing matrices.
3. The helon structure of the fermions and the resulting Koide mass relations. This helon result is very interesting because helons naturally embed into the framed braided belt networks of framed loop quantum gravity.
4. A clear result showing that there are only three generations of standard model elementary fermion states.

After all of this has been elucidated we move on to the problem of considering how quantum spacetime and gravity emerges. At this level we can also consider bosons.

3.8 The exceptional Jordan algebra and the Dirac-type equation

In this section we state a minimum of key results about exceptional Jordan algebras and the emergent Dirac-type equation explained by Dray et.al.[18] in order to set up our study into the matter in sec. (3.9).

The exceptional Jordan algebra $J_3^{\mathbb{O}}$ is the algebra $\mathcal{H}(3,\mathbb{O})$ of 3×3 hermitian matrices with octonion elements and with Jordan product defined between any two elements $A,B\in J_3^{\mathbb{O}}$ as

$$A\circ B\equiv\frac{1}{2}(AB+BA),\quad AB \text{ is given by the usual matrix multiplication.} \tag{3.122}$$

We also have $A^2\equiv A\circ A$, $A^3\equiv A^2\circ A\equiv A\circ A^2$, the Freudenthal product (with usual notion of trace)

$$A*B\equiv A\circ B-\frac{1}{2}\left(A\mathrm{Tr}(B)+B\mathrm{Tr}(A)\right)+\frac{1}{2}\left(\mathrm{Tr}(A)\mathrm{Tr}(B)-\mathrm{Tr}(A\circ B)\right)I \tag{3.123}$$

trace reversal $\tilde{A}\equiv A-\mathrm{Tr}(A)I$ and the determinant $\det(A)=\frac{1}{3}\mathrm{Tr}((A*A)\circ A)$.

The physical states are the unital trace idempotents. We defer the justification of this point to the end of this section to avoid cluttering up the following crucial conceptual flow of ideas. This subspace of $J_3^{\mathbb{O}}$ defines the octonionic projective plane $\mathbb{O}P^2$:

$$\mathbb{O}P^2\equiv\{X\in J_3^{\mathbb{O}}|X\circ X=X,\mathrm{Tr}(X)=1\}. \tag{3.124}$$

Importantly for our purposes, this turns out to be equivalent to

$$\mathbb{O}P^2=\{X\in J_3^{\mathbb{O}}|X*X=0,\mathrm{Tr}(X)=1\}. \tag{3.125}$$

Hence to find physical states we need to find X that solves the equation $X*X=0$. Suppose we have $\alpha\in\mathbb{R}$, $\Psi\in\mathbb{O}^2$ and $\mathbf{P}\in\mathcal{H}(2,\mathbb{O})$. We can write

$$P=\begin{pmatrix}\mathbf{P} & \Psi\\ \Psi^\dagger & \alpha\end{pmatrix}\in J_3^{\mathbb{O}} \tag{3.126}$$

where $\dagger:\mathbf{e}_k\mapsto-\mathbf{e}_k$ takes the octonionic conjugate of an octonion. In order for $P\in\mathbb{O}P^2\subset J_3^{\mathbb{O}}$ to be a unital trace idempotent, and hence a viable quantum state, we use the condition that the Freudenthal product must then vanish: $P*P=0$. Written out explicitly, this gives

$$P*P=\begin{pmatrix}\widetilde{\Psi\Psi^\dagger}-\alpha\tilde{\mathbf{P}} & \tilde{\mathbf{P}}\Psi\\ (\tilde{\mathbf{P}}\Psi)^\dagger & \det(\mathbf{P})\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}. \tag{3.127}$$

Note that the first matrix is still written in block form where the upper left element is actually a 2×2 matrix, the upper right and lower left elements are respectively 2×1 and 1×2 matrices and the lower right element is a 1×1 matrix. The Dirac-type momentum space-looking equation is

$$\tilde{\mathbf{P}}\Psi=0. \tag{3.128}$$

The *massless* aspect is given by the accompanying equation $\det(\mathbf{P}) = 0$. The general solution to the equation $P * P = 0$ is given by

$$\Psi = \theta\psi, \quad \mathbf{P} = \theta\theta^\dagger, \quad \alpha = |\psi|^2 \tag{3.129}$$

where $\psi \in \mathbb{O}$ (arbitrary) and $\theta \in \mathbb{O}^2$ has components which lie in the complex subalgebra $\mathbb{C} \subset \mathbb{O}$ determined by P. Thus, mathematically $\tilde{\mathbf{P}}\Psi = 0$ is equivalent to the momentum space 1+9-dimensional massless Dirac equation. In order to see this, we need to rewrite the usual Dirac equation with $\mathbb{C}^4$ spinors replaced with $\mathbb{H}^2$ spinors. The replacements are as follows:

$$\mathbb{C}^4_{\mathbf{e}_7} \ni u\left(\mathbf{p}, \frac{1}{2}\right) = A \begin{pmatrix} m_+ \\ p_+ \\ m_- \\ -p_+ \end{pmatrix} \to \begin{pmatrix} m_- - ip_+ \\ -p_+ + im_+ \end{pmatrix} \in \mathbb{H}^2_{\mathbf{e}_7,i} \tag{3.130}$$

$$\mathbb{C}^4_{\mathbf{e}_7} \ni u\left(\mathbf{p}, -\frac{1}{2}\right) = A \begin{pmatrix} p_- \\ m_- \\ -p_- \\ m_+ \end{pmatrix} \to \begin{pmatrix} -p_- - im_- \\ m_+ + ip_- \end{pmatrix} \in \mathbb{H}^2_{\mathbf{e}_7,i} \tag{3.131}$$

$$\mathbb{C}^4_{\mathbf{e}_7} \ni v\left(\mathbf{p}, \frac{1}{2}\right) = A \begin{pmatrix} p_- \\ m_- \\ p_- \\ -m_+ \end{pmatrix} \to \begin{pmatrix} p_- - im_- \\ -m_+ + ip_- \end{pmatrix} \in \mathbb{H}^2_{\mathbf{e}_7,i} \tag{3.132}$$

$$\mathbb{C}^4_{\mathbf{e}_7} \ni v\left(\mathbf{p}, -\frac{1}{2}\right) = A \begin{pmatrix} -m_+ \\ -p_+ \\ m_- \\ -p_+ \end{pmatrix} \to \begin{pmatrix} m_- + ip_+ \\ -p_+ - im_+ \end{pmatrix} \in \mathbb{H}^2_{\mathbf{e}_7,i} \tag{3.133}$$

where $A = \frac{1}{\sqrt{2(m+p^0)}}$, $m_\pm = m + p^0 \pm p^3$, $p_\pm = p^2 \pm \mathbf{e}_7 p^3$ and $\mathbf{e}_7$ is the imaginary unit generating $\mathbb{C}_{\mathbf{e}_7} \subset \mathbb{O}_{\mathbf{e}_7,\mathbf{e}_1,\mathbf{e}_2}$. The subscript i takes on the values $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_1\mathbf{e}_2 \equiv \mathbf{e}_4$ giving exactly three generations of fermionic solutions. The conversion of the usual 4×4 Dirac matrix operator to a 2×2 equivalent matrix operator is presented in the next section. See Eqn. (3.142). It can be readily checked that the operator and its trace reverse respectively annihilates the quaternionic Dirac u and v-type spinors.

3.8.1 Why is $\mathbb{O}P^2$ the state space for $J_3^{\mathbb{O}}$?

Suppose we have $A \in J_3^{\mathbb{O}}$. We wish to understand the eigenvalue equation

$$A \circ X = \lambda X \tag{3.134}$$

where λ are the eigenvalues of A and X are the eigenmatrices. The X needs to be 3×3 matrices in order for the Jordan product to be well-defined since $A \circ X = \frac{1}{2}(AX + XA)$. The eigenvalues of this equation are real: $\lambda \in \mathbb{R}$. These eigenvalues also solve the usual characteristic equation -det$(A - \lambda I_3) = 0$, which, in the case of octonionic 3×3 Jordan matrices takes the form

$$-\det(A - \lambda I_3) = \lambda^3 - (\mathrm{tr}\, A)\lambda^2 + \mathrm{tr}(A * A)\lambda - (\det A) = 0 \tag{3.135}$$

where we have the Freudenthal product

$$A * B = A \circ B - \frac{1}{2}\left(A\mathrm{tr}(B) + B\mathrm{tr}(A)\right) + \frac{1}{2}\left(\mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(A \circ B)\right) I_3 \tag{3.136}$$

and the determinant function given by

$$\det(A) = \frac{1}{3}\mathrm{tr}\left((A * A) \circ A\right). \tag{3.137}$$

The Jordan and Freudenthal products can be thought of in some sense as generalizations of the usual dot and cross products. Given non-degenerate eigenvalues, the normalized eigenmatrices are primitive idempotents. Every $A \in J_3^{\mathbb{O}}$ can be diagonalized, and so a solution to the eigenmatrix problem for a specific eigenvalue, when normalized, corresponds to an idempotent of trace 1. The set of trace 1 idempotents defines the octonionic projective plane:

$$\begin{aligned} \mathbb{O}P^2 &\equiv \{X \in J_3^{\mathbb{O}} = \mathcal{H}(3, \mathbb{O}) : X \circ X = X,\ \ \mathrm{tr}X = 1\} && (3.138) \\ &= \{X \in J_3^{\mathbb{O}} : X * X = 0,\ \ \mathrm{tr}X = 1\}. && (3.139) \end{aligned}$$

3.9 Helons and mass

First, we write an octonion $p \in \mathbb{O}_{\mathbf{e}_7,\mathbf{e}_1,\mathbf{e}_2}$ as

$$p = p_1 + p^2\mathbf{e}_7 + p^3\mathbf{e}_2 + p^4\mathbf{e}_3 + p^5\mathbf{e}_4 + p^6\mathbf{e}_5 + p^7\mathbf{e}_6 + p^8\mathbf{e}_1 \tag{3.140}$$

with multiplication rules $\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_4$, $\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_7$, $\mathbf{e}_1\mathbf{e}_5 = \mathbf{e}_6$, $\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_5$, $\mathbf{e}_2\mathbf{e}_6 = \mathbf{e}_7$, $\mathbf{e}_3\mathbf{e}_4 = \mathbf{e}_6$ and $\mathbf{e}_4\mathbf{e}_5 = \mathbf{e}_7$.

Compare/contrast the usual Dirac equation (written in quaternionic form, and replacing the spacetime index 3 with the label 9) with the momentum space octonionic Dirac-type equation, respectively listed here:

$$\begin{pmatrix} -p^0 + p^9 & p^1 - p^2\mathbf{e}_7 - m\mathbf{e}_1 \\ p^1 + p^2\mathbf{e}_7 + m\mathbf{e}_1 & -p^0 - p^9 \end{pmatrix} \psi = 0. \tag{3.141}$$

$$\begin{pmatrix} -p^0+p^9 & p^1-p^2\mathbf{e}_7-p^3\mathbf{e}_2-p^4\mathbf{e}_3-p^5\mathbf{e}_4-p^6\mathbf{e}_5-p^7\mathbf{e}_6-p^8\mathbf{e}_1 \\ p^1+p^2\mathbf{e}_7+\cdots+p^8\mathbf{e}_1 & -p^0-p^9 \end{pmatrix} \psi = 0. \tag{3.142}$$

We can rewrite the above octonion as

$$(p^1 + p^2\mathbf{e}_7)1 + (p^3 + p^5\mathbf{e}_7)\mathbf{e}_1 + (p^4 + p^8\mathbf{e}_7)\mathbf{e}_2 + (p^6 + p^7\mathbf{e}_7)\mathbf{e}_4 \in \mathbb{C}_{\mathbf{e}_7} \oplus \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2} \tag{3.143}$$

as vector spaces. The multiplication laws are inherited from $\mathbb{O}_{\mathbf{e}_7,\mathbf{e}_1,\mathbf{e}_2}$. With obvious relabellings we can write this as

$$q = w + x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_4 \in \mathbb{C}_{\mathbf{e}_7} \oplus \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}, \quad w, x, y, z \in \mathbb{C}_{\mathbf{e}_7}. \tag{3.144}$$

Solutions to the Dirac equation come in three generations, which we can enumerate as

1. Generation 1: $\mathbb{H}^2_{\mathbf{e}_7,\mathbf{e}_1}$
2. Generation 2: $\mathbb{H}^2_{\mathbf{e}_7,\mathbf{e}_2}$
3. Generation 3: $\mathbb{H}^2_{\mathbf{e}_7,\mathbf{e}_4}$.

The mass term however is contained in the pure part of $\mathbb{C}_{\mathbf{e}_7} \oplus \mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$. The quaternion algebra $\mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$ is distinct from the three physical solutions built on $\mathbb{H}_{\mathbf{e}_7,\mathbf{e}_1}$, $\mathbb{H}_{\mathbf{e}_7,\mathbf{e}_2}$ and $\mathbb{H}_{\mathbf{e}_7,\mathbf{e}_4}$. If we just took the magnitude of the mass part:

$$\text{Mass Part}: \; x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_4 \tag{3.145}$$

we should not expect to get the physical mass of a particular particle that sits inside one well-defined generation of fermionic particles. To get physical masses, we would have to project the overall mass part down to an $\mathbf{e}_1$ part, an $\mathbf{e}_2$ part or an $\mathbf{e}_4$ part. We can therefore write three general projection operators P_R, P_G and P_B by defining three equilateral triangle defining unit length three-dimensional vectors $\mathbf{r}$, $\mathbf{g}$ and $\mathbf{b}$ such that $\mathbf{r}+\mathbf{g}+\mathbf{b}=0$, taking $\boldsymbol{\sigma} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4)$ and writing

$$P_R = \frac{1}{2}(1+\mathbf{r}\cdot\boldsymbol{\sigma}), \quad P_G = \frac{1}{2}(1+\mathbf{g}\cdot\boldsymbol{\sigma}), \quad P_B = \frac{1}{2}(1+\mathbf{b}\cdot\boldsymbol{\sigma}). \tag{3.146}$$

One thing that can be immediately noticed with these three projection operators is that they braid each other:

$$P_RP_GP_R = P_GP_RP_G, \quad P_RP_BP_R = P_BP_RP_B, \quad P_GP_BP_G = P_BP_GP_B. \tag{3.147}$$

These projections onto physical masses generate the circular Artin braid group B_3^c. It stands to reason then that it should be possible to represent one generation of standard model fermion states by elements of B_3^c. However the coefficients are elements of $\mathbb{C}_{\mathbf{e}_7}$ and these elements do not commute with the elements of $\mathbb{H}_{\mathbf{e}_1,\mathbf{e}_2}$. The projection on to the "w" part generates the braid group B_2. This does not commute with B_3^c but has a semi-direct product structure so that we can think of a fermionic matter state as being a braided ribbon structure with twisted ribbons. Such a model, totally disconnected from all of the Jordan algebra background motivation given here, has been independently developed and is known as the helon model. The helon model seems to provide a promising way of embedding fermionic matter states into the braided belt networks of framed loop quantum gravity[8, 6, 7]. The point of view coming from framed loop quantum gravity would then be that the untwisted structure of other spin networks gives quantum spacetime, while the extra twist degree of freedom allowed on the braidings naturally allows various embedded twisted braided structures to be identified with fermionic matter states, thus unifying everything in physics in terms of the fundamental braided belt foam.

We can check idempotency of the projections as follows.

$$P_RP_R = \frac{1}{4}(1+\mathbf{r}\cdot\boldsymbol{\sigma})(1+\mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{4}(1+2\mathbf{r}\cdot\boldsymbol{\sigma}+(\mathbf{r}\cdot\boldsymbol{\sigma})(\mathbf{r}\cdot\boldsymbol{\sigma})). \tag{3.148}$$

But

$$(\mathbf{r}\cdot\boldsymbol{\sigma})(\mathbf{r}\cdot\boldsymbol{\sigma}) = (\mathbf{r}\cdot\boldsymbol{\sigma}'\mathbf{e}_7)(\mathbf{r}\cdot\boldsymbol{\sigma}'\mathbf{e}_7) = -(\mathbf{r}\cdot\boldsymbol{\sigma}')(\mathbf{r}\cdot\boldsymbol{\sigma}')\mathbf{e}_7^2 = 1$$

where $\boldsymbol{\sigma}'$ has the same algebraic properties as the Pauli algebra. Hence $P_R^2 = P_R$, and is therefore a projection. It appears then that fermion mass states are twisted belt braidings. This gives rise to the helon model, *except here, it becomes clear how to extend from one generation to three generations.*

The Helon model gives one generation of standard model fermions. Here we extend to all three generations:[2]

$$\begin{aligned}
\nu_e &= [0,0,0](\sigma_{2,1}^{-1}\sigma_{1,1}), & \bar{\nu}_e = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,0,0], && (3.149)\\
\bar{d}^{\bar{r}} &= [0,0,1](\sigma_{2,1}^{-1}\sigma_{1,1}), & d^r = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,0,-1], && (3.150)\\
\bar{d}^{\bar{y}} &= [0,1,0](\sigma_{2,1}^{-1}\sigma_{1,1}), & d^y = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,-1,0], && (3.151)\\
\bar{d}^{\bar{b}} &= [1,0,0](\sigma_{2,1}^{-1}\sigma_{1,1}), & d^b = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,0,0], && (3.152)\\
u^r &= [0,1,1](\sigma_{2,1}^{-1}\sigma_{1,1}), & \bar{u}^{\bar{r}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,-1,-1], && (3.153)\\
u^y &= [1,1,0](\sigma_{2,1}^{-1}\sigma_{1,1}), & \bar{u}^{\bar{y}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,-1,0], && (3.154)\\
u^b &= [1,0,1](\sigma_{2,1}^{-1}\sigma_{1,1}), & \bar{u}^{\bar{b}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,0,-1], && (3.155)\\
e^+ &= [1,1,1](\sigma_{2,1}^{-1}\sigma_{1,1}), & e^- = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,-1,-1]. && (3.156)
\end{aligned}$$

$$\begin{aligned}
\nu_\mu &= [0,0,0](\sigma_{2,2}^{-1}\sigma_{1,2}), & \bar{\nu}_\mu = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,0,0], && (3.157)\\
\bar{s}^{\bar{r}} &= [0,0,1](\sigma_{2,2}^{-1}\sigma_{1,2}), & s^r = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,0,-1], && (3.158)\\
\bar{s}^{\bar{y}} &= [0,1,0](\sigma_{2,2}^{-1}\sigma_{1,2}), & s^y = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,-1,0], && (3.159)\\
\bar{s}^{\bar{b}} &= [1,0,0](\sigma_{2,2}^{-1}\sigma_{1,2}), & s^b = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,0,0], && (3.160)\\
c^r &= [0,1,1](\sigma_{2,2}^{-1}\sigma_{1,2}), & \bar{c}^{\bar{r}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,-1,-1], && (3.161)\\
c^y &= [1,1,0](\sigma_{2,2}^{-1}\sigma_{1,2}), & \bar{c}^{\bar{y}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,-1,0], && (3.162)\\
c^b &= [1,0,1](\sigma_{2,2}^{-1}\sigma_{1,2}), & \bar{c}^{\bar{b}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,0,-1], && (3.163)\\
\mu^+ &= [1,1,1](\sigma_{2,2}^{-1}\sigma_{1,2}), & \mu^- = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,-1,-1]. && (3.164)
\end{aligned}$$

$$\begin{aligned}
\nu_\tau &= [0,0,0](\sigma_{2,3}^{-1}\sigma_{1,3}), & \bar{\nu}_\tau = (\sigma_{1,3}^{-1}\sigma_{2,3})[0,0,0], && (3.165)\\
\bar{b}^{\bar{r}} &= [0,0,1](\sigma_{2,3}^{-1}\sigma_{1,3}), & b^r = (\sigma_{1,3}^{-1}\sigma_{2,3})[0,0,-1], && (3.166)\\
\bar{b}^{\bar{y}} &= [0,1,0](\sigma_{2,3}^{-1}\sigma_{1,3}), & b^y = (\sigma_{1,3}^{-1}\sigma_{2,3})[0,-1,0], && (3.167)\\
\bar{b}^{\bar{b}} &= [1,0,0](\sigma_{2,3}^{-1}\sigma_{1,3}), & b^b = (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,0,0], && (3.168)\\
t^r &= [0,1,1](\sigma_{2,3}^{-1}\sigma_{1,3}), & \bar{t}^{\bar{r}} = (\sigma_{1,3}^{-1}\sigma_{2,3})[0,-1,-1], && (3.169)\\
t^y &= [1,1,0](\sigma_{2,3}^{-1}\sigma_{1,3}), & \bar{t}^{\bar{y}} = (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,-1,0], && (3.170)\\
t^b &= [1,0,1](\sigma_{2,3}^{-1}\sigma_{1,3}), & \bar{t}^{\bar{b}} = (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,0,-1], && (3.171)\\
\tau^+ &= [1,1,1](\sigma_{2,3}^{-1}\sigma_{1,3}), & \tau^- = (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,-1,-1]. && (3.172)
\end{aligned}$$

The first column above consists purely of weak isospin up states whereas the second column above consists purely of weak isospin down states. These are all elements of $(B_3^c) \ltimes (B_2)^3$. The above σ

[2]Note that due to the restrictions to associative quaternionic subalgebras of the octonions, we do not have to worry about the nonassociativity of the octonions.

notation is defined for the three generations as follows:

Generation 1. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_1} \subset B_7^c$. Octonion generators: $\mathbf{e}_7, \mathbf{e}_1$. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_1} \subset B_7^c$ generators:

$$\sigma_{1,1} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_7), \quad \sigma_{2,1} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_1), \quad \sigma_{3,1} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_3). \tag{3.173}$$

Generation 2. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_2} \subset B_7^c$. Octonion generators: $\mathbf{e}_7, \mathbf{e}_2$. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_2} \subset B_7^c$ generators:

$$\sigma_{1,2} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_7), \quad \sigma_{2,2} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_2), \quad \sigma_{3,2} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_6). \tag{3.174}$$

Generation 3. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_4} \subset B_7^c$. Octonion generators: $\mathbf{e}_7, \mathbf{e}_4$. $(B_3^c)_{\mathbf{e}_7,\mathbf{e}_4} \subset B_7^c$ generators:

$$\sigma_{1,3} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_7), \quad \sigma_{2,3} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_4), \quad \sigma_{3,3} = \frac{1}{\sqrt{2}}(1+\mathbf{e}_5). \tag{3.175}$$

By inspection we see that there is only one element of $(B_7^c)_{\mathbf{e}_7,\mathbf{e}_1,\mathbf{e}_2}$ that is held in common in all three generations: σ_1. All other generators only appear once with no overlaps over the generations. Sequences of σ's are called braid words. Elements $[a,b,c] \in (B_2)^3$ are called twist words. If the braid word is trivial we have a pure twist word and if the twist word is trivial, we have a pure braid word. For more details concerning twists, braids and their nontrivial relationships, see [23].

We can choose another representation for the particles. Even though the chiral states in the helon model are represented in terms of (twisted) braid crossings, the states can be mapped onto the permutation group, and hence one generation can be represented by 3×3 matrices. We can think of each generation as living inside a certain 3×3 block within the set of 7×7 matrices. If we use the $\mathbf{e}_i$ octonion notation and think of the columns and rows being numbered as $7, 1, 2, 3, 4, 5, 6$ then

1. The generation one helons form a $(7, 1, 3)$ block.

2. The generation two helons form a $(7, 2, 6)$ block.

3. The generation three helons form a $(7, 4, 5)$ block.

In these 3×3 blocks we can take the identity matrix I_3 and associate $(I_3)_{1,1} = 1$ with the top of the first strand, $(I_3)_{2,2} = 1$ with the top of the second strand and $(I_3)_{3,3} = 1$ with the top of the third strand. The positions the strands get permuted to at the bottom of a helon diagram determines how the nonzero elements of I_3 get permuted. The permutations act horizontally for negatively charged states and vertically for positively charged states. The neutrinos are permuted horizontally. The action of charge on these matrices is by multiplying individual nonzero entries of various matrices by ω or $\overline{\omega}$ where $\omega = e^{\mathbf{e}_7 2\pi/3}$.

The first generation of fermion states can thus be represented as

$$\nu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_L^- = \overline{\omega} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$e_R^+ = \omega \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_L^+ = \omega \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad e_R^- = \overline{\omega} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$u_L^r = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}, \quad u_L^g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega & 0 & 0 \end{pmatrix}, \quad u_L^b = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix},$$

$$\bar{d}_L^{\bar{r}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}, \quad \bar{d}_L^{\bar{g}} = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{d}_L^{\bar{b}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix},$$

$$\bar{u}_L^{\bar{r}} = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{u}_L^{\bar{g}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \end{pmatrix}, \quad \bar{u}_L^{\bar{b}} = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \end{pmatrix},$$

$$d_L^r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \end{pmatrix}, \quad d_L^g = \begin{pmatrix} 0 & \overline{\omega} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad d_L^b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \end{pmatrix},$$

$$u_R^r = \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \quad u_R^g = \begin{pmatrix} 0 & 0 & \omega \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad u_R^b = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$$

$$\bar{d}_R^{\bar{r}} = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \bar{d}_R^{\bar{g}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \quad \bar{d}_R^{\bar{b}} = \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\bar{u}_R^{\bar{r}} = \begin{pmatrix} 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix} \quad \bar{u}_R^{\bar{g}} = \begin{pmatrix} 0 & 0 & \overline{\omega} \\ \overline{\omega} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \bar{u}_R^{\bar{b}} = \begin{pmatrix} 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix}$$

$$d_R^r = \begin{pmatrix} 0 & 0 & 1 \\ \overline{\omega} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad d_R^g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix} \quad d_R^b = \begin{pmatrix} 0 & 0 & \overline{\omega} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice at this point that the lepton matrices are circulant. This naturally leads to the Koide mass matrices [75].

At this point we make some remarks relating to the CKM and $PMNS$ matrices of the standard model before resuming our main discussion about fermions as helons. We have seen that the physical mass term formed the pure part of the algebras $(\mathbb{H}_{\mathbf{e}_7,i}, +, \cdot; \mathbb{C}_{\mathbf{e}_7})$ for $i = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$, and that we could form the general projections given by Eqns. (3.146). The projections as we have presented them are more general than what is required to actually project precisely onto mass eigenstates. It could be that the projections are related to three generations of mass eigenstates by a transformation $\mathbf{U} \in U(3)$. If $\mathbf{U} \neq I_3$ then we have nontrivial linear combinations of physical mass eigenstates, rather than just the pure mass eigenstates themselves. Hence we also have a natural theoretical origin for the $PMNS$ and CKM matrices of the standard model, although more work would need to be done to determine if a prediction could be made concerning the numerical values of the $PMNS$ and CKM matrix elements. In [75] evidence is presented that circulant matrices, Koide mass formulas and the values of the $PMNS$ and CKM matrix elements are all related. We defer a more detailed discussion of the Koide mass relations to the end of this section.

Transitioning back to our helon discussion:

As 3×3 blocks the other two generations of fermion states look identical. The difference can be seen in the way these block matrices embed in $\mathrm{Mat}(7, \mathbb{C}_{\mathbf{e}_7})$. For example:

$$\bar{d}_L^{\bar{g}} = \begin{pmatrix} 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{s}_L^{\bar{g}} = \begin{pmatrix} 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.176}$$

$$\bar{b}_L^{\bar{g}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.177}$$

It would be interesting to take these matrices and try to reconstruct what the corresponding $(B_2)^6 \rtimes B_6^c$ representations are.

The 3×3 block representation for the $W^{\pm}$ helons are $W^- = \varpi I_3$ and $W^+ = \omega I_3$. The photon is the identity $\gamma = I_3$. The Z^0 would have to be some linear combination of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \varpi \end{pmatrix}, \quad \begin{pmatrix} \varpi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.178}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varpi & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi \end{pmatrix}, \quad \begin{pmatrix} \varpi & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.179}$$

In [7] it was shown that

$$(B_2)^3 \ltimes B_3^c \ni ([a, b, c], \Lambda) = [a', b', c'] \in (B_2)^3. \tag{3.180}$$

In other words it was shown that a twisted braid can always be rewritten in pure twist form with trivial braiding. As explicitly given in [8], the braid generators of the circular Artin braid group B_3^c can be written as twist vectors as follows, which we express here, but with all three generations rather than just one with $a = 1, 2, 3$:

$$\begin{aligned}
\sigma_{1,a} &= \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right], \qquad \sigma_{1,a}^{-1} = \left[-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right], && (3.181)\\
\sigma_{2,a} &= \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \qquad \sigma_{2,a}^{-1} = \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right], && (3.182)\\
\sigma_{3,a} &= \left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right], \qquad \sigma_{3,a}^{-1} = \left[-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]. && (3.183)
\end{aligned}$$

The important question at this point is whether twisted braids can be written as pure braid words with trivial twisting. The answer in general is no. However, the answer is yes in some cases and happens to be yes for the particular cases given by the standard model matter state twisted braid representations above. The representations are not unique but nevertheless this result means that the states above can actually be written as pure braid elements of B_3^c. This can be done, here for all three generations, by noting from [23] that:

$$\begin{aligned}
[1,0,0] &= [0,0,0](\sigma_{2,a}\sigma_{3,a}), \quad [-1,-1,-1] = [0,0,0](\sigma_{2,a}^{-1}\sigma_{3,a}^{-1})(\sigma_{3,a}^{-1}\sigma_{1,a}^{-1})(\sigma_{1,a}^{-1}\sigma_{2,a}^{-1}),\\
[0,1,0] &= [0,0,0](\sigma_{3,a}\sigma_{1,a}), \quad [0,-1,-1] = [0,0,0](\sigma_{2,a}^{-1}\sigma_{3,a}^{-1})(\sigma_{3,a}^{-1}\sigma_{1,a}^{-1}),\\
[0,0,1] &= [0,0,0](\sigma_{1,a}\sigma_{2,a}), \quad [-1,-1,0] = [0,0,0](\sigma_{1,a}^{-1}\sigma_{2,a}^{-1})(\sigma_{2,a}^{-1}\sigma_{3,a}^{-1}),\\
[1,0,1] &= [0,0,0](\sigma_{3,a}\sigma_{1,a})(\sigma_{1,a}\sigma_{2,a}), \quad [-1,0,-1] = [0,0,0](\sigma_{3,a}^{-1}\sigma_{1,a}^{-1})(\sigma_{1,a}^{-1}\sigma_{2,a}^{-1}),\\
[1,1,0] &= [0,0,0](\sigma_{1,a}\sigma_{2,a})(\sigma_{2,a}\sigma_{3,a}), \quad [0,0,-1] = [0,0,0](\sigma_{1,a}^{-1}\sigma_{2,a}^{-1}),\\
[0,1,1] &= [0,0,0](\sigma_{2,a}\sigma_{3,a})(\sigma_{3,a}\sigma_{1,a}), \quad [0,-1,0] = [0,0,0](\sigma_{3,a}^{-1}\sigma_{1,a}^{-1}),\\
[1,1,1] &= [0,0,0](\sigma_{2,a}\sigma_{3,a})(\sigma_{3,a}\sigma_{1,a})(\sigma_{1,a}\sigma_{2,a}), \quad [-1,0,0] = [0,0,0](\sigma_{2,a}^{-1}\sigma_{3,a}^{-1}).
\end{aligned} \tag{3.184}$$

Hence the pure braid word form belonging entirely within B_3^c for the three generations of standard model elementary fermionic matter states can be given by

$$\nu_e = [0,0,0](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.185}$$
$$\bar{d}^{\bar{r}} = [0,0,1](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{1,1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.186}$$
$$\bar{d}^{\bar{g}} = [0,1,0](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{3,1}\sigma_{1,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.187}$$
$$\bar{d}^{\bar{b}} = [1,0,0](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{2,1}\sigma_{3,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.188}$$
$$u^r = [0,1,1](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{2,1}\sigma_{3,1})(\sigma_{3,1}\sigma_{1,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.189}$$
$$u^g = [1,1,0](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{1,1}\sigma_{2,1})(\sigma_{2,1}\sigma_{3,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.190}$$
$$u^b = [1,0,1](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{3,1}\sigma_{1,1})(\sigma_{1,1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.191}$$
$$e^+ = [1,1,1](\sigma_{2,1}^{-1}\sigma_{1,1}) = (\sigma_{2,1}\sigma_{3,1})(\sigma_{3,1}\sigma_{1,1})(\sigma_{1,1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{1,1}), \tag{3.192}$$
$$\bar{\nu}_e = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,0,0] = (\sigma_{1,1}^{-1}\sigma_{2,1}), \tag{3.193}$$
$$d^r = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,0,-1] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{1,1}^{-1}\sigma_{2,1}^{-1}), \tag{3.194}$$
$$d^g = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,-1,0] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{3,1}^{-1}\sigma_{1,1}^{-1}), \tag{3.195}$$
$$d^b = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,0,0] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{3,1}^{-1}), \tag{3.196}$$
$$\bar{u}^{\bar{r}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[0,-1,-1] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{3,1}^{-1})(\sigma_{3,1}^{-1}\sigma_{1,1}^{-1}), \tag{3.197}$$
$$\bar{u}^{\bar{g}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,-1,0] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{1,1}^{-1}\sigma_{2,1}^{-1})(\sigma_{2,1}^{-1}\sigma_{3,1}^{-1}), \tag{3.198}$$
$$\bar{u}^{\bar{b}} = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,0,-1] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{3,1}^{-1}\sigma_{1,1}^{-1})(\sigma_{1,1}^{-1}\sigma_{2,1}^{-1}), \tag{3.199}$$
$$e^- = (\sigma_{1,1}^{-1}\sigma_{2,1})[-1,-1,-1] = (\sigma_{1,1}^{-1}\sigma_{2,1})(\sigma_{2,1}^{-1}\sigma_{3,1}^{-1})(\sigma_{3,1}^{-1}\sigma_{1,1}^{-1})(\sigma_{1,1}^{-1}\sigma_{2,1}^{-1}). \tag{3.200}$$
$$\nu_\mu = [0,0,0](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.201}$$
$$\bar{s}^{\bar{r}} = [0,0,1](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{1,2}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.202}$$
$$\bar{s}^{\bar{g}} = [0,1,0](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{3,2}\sigma_{1,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.203}$$
$$\bar{s}^{\bar{b}} = [1,0,0](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{2,2}\sigma_{3,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.204}$$
$$c^r = [0,1,1](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{2,2}\sigma_{3,2})(\sigma_{3,2}\sigma_{1,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.205}$$
$$c^g = [1,1,0](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{1,2}\sigma_{2,2})(\sigma_{2,2}\sigma_{3,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.206}$$
$$c^b = [1,0,1](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{3,2}\sigma_{1,2})(\sigma_{1,2}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.207}$$
$$\mu^+ = [1,1,1](\sigma_{2,2}^{-1}\sigma_{1,2}) = (\sigma_{2,2}\sigma_{3,2})(\sigma_{3,2}\sigma_{1,2})(\sigma_{1,2}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{1,2}), \tag{3.208}$$
$$\bar{\nu}_\mu = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,0,0] = (\sigma_{1,2}^{-1}\sigma_{2,2}), \tag{3.209}$$
$$s^r = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,0,-1] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{1,2}^{-1}\sigma_{2,2}^{-1}), \tag{3.210}$$
$$s^g = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,-1,0] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{3,2}^{-1}\sigma_{1,2}^{-1}), \tag{3.211}$$
$$s^b = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,0,0] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{3,2}^{-1}), \tag{3.212}$$
$$\bar{c}^{\bar{r}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[0,-1,-1] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{3,2}^{-1})(\sigma_{3,2}^{-1}\sigma_{1,2}^{-1}), \tag{3.213}$$
$$\bar{c}^{\bar{g}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,-1,0] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{1,2}^{-1}\sigma_{2,2}^{-1})(\sigma_{2,2}^{-1}\sigma_{3,2}^{-1}), \tag{3.214}$$
$$\bar{c}^{\bar{b}} = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,0,-1] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{3,2}^{-1}\sigma_{1,2}^{-1})(\sigma_{1,2}^{-1}\sigma_{2,2}^{-1}), \tag{3.215}$$
$$\mu^- = (\sigma_{1,2}^{-1}\sigma_{2,2})[-1,-1,-1] = (\sigma_{1,2}^{-1}\sigma_{2,2})(\sigma_{2,2}^{-1}\sigma_{3,2}^{-1})(\sigma_{3,2}^{-1}\sigma_{1,2}^{-1})(\sigma_{1,2}^{-1}\sigma_{2,2}^{-1}). \tag{3.216}$$

$$
\begin{aligned}
\nu_\tau &= [0,0,0](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.217)\\
\bar{b}^{\bar{r}} &= [0,0,1](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{1,3}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.218)\\
\bar{b}^{\bar{g}} &= [0,1,0](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{3,3}\sigma_{1,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.219)\\
\bar{b}^{\bar{b}} &= [1,0,0](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{2,3}\sigma_{3,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.220)\\
t^{r} &= [0,1,1](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{2,3}\sigma_{3,3})(\sigma_{3,3}\sigma_{1,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.221)\\
t^{g} &= [1,1,0](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{1,3}\sigma_{2,3})(\sigma_{2,3}\sigma_{3,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.222)\\
t^{b} &= [1,0,1](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{3,3}\sigma_{1,3})(\sigma_{1,3}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.223)\\
\tau^{+} &= [1,1,1](\sigma_{2,3}^{-1}\sigma_{1,3}) = (\sigma_{2,3}\sigma_{3,3})(\sigma_{3,3}\sigma_{1,3})(\sigma_{1,3}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{1,3}), && (3.224)\\
\bar{\nu}_\tau &= (\sigma_{1,3}^{-1}\sigma_{2,3})[0,0,0] = (\sigma_{1,3}^{-1}\sigma_{2,3}), && (3.225)\\
b^{r} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[0,0,-1] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{1,3}^{-1}\sigma_{2,3}^{-1}), && (3.226)\\
b^{g} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[0,-1,0] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{3,3}^{-1}\sigma_{1,3}^{-1}), && (3.227)\\
b^{b} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,0,0] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{3,3}^{-1}), && (3.228)\\
\bar{t}^{\bar{r}} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[0,-1,-1] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{3,3}^{-1})(\sigma_{3,3}^{-1}\sigma_{1,3}^{-1}), && (3.229)\\
\bar{t}^{\bar{g}} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,-1,0] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{1,3}^{-1}\sigma_{2,3}^{-1})(\sigma_{2,3}^{-1}\sigma_{3,3}^{-1}), && (3.230)\\
\bar{t}^{\bar{b}} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,0,-1] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{3,3}^{-1}\sigma_{1,3}^{-1})(\sigma_{1,3}^{-1}\sigma_{2,3}^{-1}), && (3.231)\\
\tau^{-} &= (\sigma_{1,3}^{-1}\sigma_{2,3})[-1,-1,-1] = (\sigma_{1,3}^{-1}\sigma_{2,3})(\sigma_{2,3}^{-1}\sigma_{3,3}^{-1})(\sigma_{3,3}^{-1}\sigma_{1,3}^{-1})(\sigma_{1,3}^{-1}\sigma_{2,3}^{-1}). && (3.232)
\end{aligned}
$$

This extends the one generation results of [23]. The Helon model naturally embeds into the braided belt networks of framed loop quantum gravity. Here we point out the connection between the Helon model, expressed immediately above in terms of pure braid words, and the minimal left ideal states of $\mathrm{End}(\mathbb{C}\otimes\mathbb{O})\cong\mathbb{C}\ell(6)$. With obvious extension of further results in [23], by making the identifications

$$\sigma_{2,a}^{-1}\sigma_{1,a} = \omega_a\omega_a^\dagger, \quad (\sigma_{1,a}\sigma_{2,a}) = \alpha_{1,a}^\dagger, \quad (\sigma_{3,a}\sigma_{1,a}) = \alpha_{2,a}^\dagger, \quad (\sigma_{2,a}\sigma_{3,a}) = \alpha_{3,a}^\dagger \qquad (3.233)$$

where $a = 1,2,3$ and substituting into the sixteen equations immediately above, we recover the fermionic states given in Furey's one generation minimal left ideal based model[22, 21]. This confirms that all that we have done is in harmony with loop quantum gravity.

3.9.1 How calculations work with twisted braids

See [23] for more details on this topic. The Artin braid group B_n is what one gets if one takes the symmetric group S_n and removes the condition that each generator must square to unity. The braid group B_n is represented by n strands, generated by $n-1$ elements $\{\sigma_1,\cdots,\sigma_{n-1}\}$ which satisfy the following conditions:

$$
\begin{aligned}
\sigma_i\sigma_j &= \sigma_j\sigma_i, \qquad \forall\ |i-j|>1, && (3.234)\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i=1,\cdots,n-2. && (3.235)
\end{aligned}
$$

The framed braid group has each strand thought of as being thickened to a ribbon (so one could think of a framed braid as some restricted class of braidings in a braid group of twice the number of dimensions). Ribbons may be twisted. This gives rise to n twist operators $t_1, \cdots, t_n$ which related to each other and to the braid generators as

$$\begin{aligned} t_i t_j &= t_j t_i, \qquad \forall i, j, & (3.236)\\ \sigma_i t_j &= t_{\sigma_i(j)} \sigma_i, & (3.237) \end{aligned}$$

where $\sigma_i(j)$ is the braid where the permutation produced by σ_i has induced (j). Inverse braids are anti-automorphisms which are represented by vertical reflections of the corresponding braid representation. As already seen, the twisting and braidings of ribbons are non-commutative. There is also the notion of a circular Artin braid group B_n^c which closes the picture by allowing the $n-1$-th string to braid to its right with the first braid string. One can think of this geometrically as a top disk and a bottom disk connected by the braid strings/ribbons which are positioned around the edges of the disks, loosely mimicking a cylindrical looking object.

We have the semi-direct product $(B_n^c) \ltimes (B_2)^n$. We also have $B_2 \cong \frac{1}{2}\mathbb{Z}$. A general element $\Lambda \in B_n^c$ is called a braid word and a general element $[a_1, \cdots, a_n] \in (\frac{1}{2}\mathbb{Z})^n \cong B_2^n$ is called a twist word.

Specializing to the particular group $(B_3^c) \ltimes (B_2)^3$, the group element is denoted $([a_1, a_2, a_3], \Lambda)$ and the semi-direct product group composition law takes the form

$$\begin{aligned} & ([a_1, a_2, a_3], \Lambda_1)([b_1, b_2, b_3], \Lambda_2) \\ = \; & (P_{\Lambda_1}([b_1, b_2, b_3]) + [a_1, a_2, a_3], \Lambda_1 \Lambda_2) \\ = \; & ([b_{\pi(\Lambda_2)(1)}, b_{\pi(\Lambda_2)(2)}, b_{\pi(\Lambda_2)(3)}] + [a_1, a_2, a_3], \Lambda_1 \Lambda_2) \\ = \; & ([b_{\pi(\Lambda_2)(1)} + a_1, b_{\pi(\Lambda_2)(2)} + a_2, b_{\pi(\Lambda_2)(3)} + a_3], \Lambda_1 \Lambda_2). \end{aligned}$$

Here $P_{\Lambda_i}([...])$ is the permutation induced on $[...]$ by the braid word Λ_i. Moreover $\pi : B_3^c \to S_3$, and here in particular we have $\pi(\sigma_1) = (12)$, $\pi(\sigma_2) = (23)$ and $\pi(\sigma_3) = (31)$. With these combined twist-braid words, the ordering is such that by convention the twistings are at the tops of the framed strands and the braidings are always below the twistings. As an example, we make use the illustration in [23, p.13]:

$$\begin{aligned} [0,1,0]\sigma_1\sigma_2 &= \left(P_{\sigma_1}[0,1,0] + \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\right)\sigma_2 \\ &= \left([1,0,0] + \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]\right)\sigma_2 \\ &= \left[\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right]\sigma_2 \\ &= P_{\sigma_2}\left[\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right] + \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], \\ &= \left[\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right] + \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right] \\ &= [1,0,1]. \end{aligned} \qquad (3.238)$$

3.9.2 The Koide mass relations

Now we consider mass. Physical mass is something which can be observed (even at rest). It seems natural to then write the mass state as[3]

$$M = |M\rangle \langle M| \quad \in \text{Mat}(3, \mathbb{C}), \tag{3.239}$$

which is obviously an idempotent: $M^2 = M$. Moreover, from the Dirac equation we see that the mass terms mix up left-handed and right-handed states. Hence it also seems natural to write[4]

$$m^2 = \langle R| M |L\rangle \langle L| M |R\rangle = \langle R|M\rangle \langle M|L\rangle \langle L|M\rangle \langle M|R\rangle . \tag{3.240}$$

Mass treats left-handed and right-handed components the same way so it seems reasonable to conclude that

$$\langle M|R\rangle = \langle M|L\rangle = \pm\sqrt{m}. \tag{3.241}$$

In the helon section we see that lepton states are given in terms of circulant matrices. For leptons we can thus write

$$\sqrt{M} = \frac{\sqrt{\mu}}{r} \begin{pmatrix} r & e^{\mathbf{e}_7\delta} & e^{-\mathbf{e}_7\delta} \\ e^{-\mathbf{e}_7\delta} & r & e^{\mathbf{e}_7\delta} \\ e^{\mathbf{e}_7\delta} & e^{-\mathbf{e}_7\delta} & r \end{pmatrix}. \tag{3.242}$$

Here μ is a dimensionful scale. The eigenvalues are given by

$$\sqrt{m_j} = \sqrt{\mu}\lambda_j, \quad \lambda_j = 1 + \frac{2}{r}\cos(\delta + \omega^j), \quad j = 1, 2, 3. \tag{3.243}$$

The matrix $\sqrt{M}$ is diagonalized by the Fourier transform F_3. Thus the determinant is

$$\det(\sqrt{M}) = \lambda_1\lambda_2\lambda_3 = (\sqrt{\mu})^3(r^3 - 3r + 2\cos(3\delta)). \tag{3.244}$$

Setting $\det(\sqrt{M}) = 0$ with $\sqrt{\mu} \neq 0$ yields four solutions:

$$(r_1, \delta_1) = (\sqrt{3}, \pm\frac{\pi}{6}), \qquad (r_2, \delta_2) = (\sqrt{2}, \pm\frac{\pi}{12}). \tag{3.245}$$

Now we take the trace:

$$\text{tr}\sqrt{M} = \lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{\sqrt{\mu}}(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}). \tag{3.246}$$

But we also have

$$\text{tr}(M) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \frac{1}{\mu}(m_1 + m_2 + m_3). \tag{3.247}$$

Hence

$$\frac{(\text{Tr}(\sqrt{M}))^2}{\text{Tr}(M)} = \frac{1}{\mu}\mu\frac{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}{m_1 + m_2 + m_3} = \frac{(\lambda_1 + \lambda_2 + \lambda_3)^2}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \tag{3.248}$$

[3]This identification of Mat$(3, \mathbb{C})$ is not quite right because the mass term in the Dirac operator has noncommutativity between the $\mathbf{e}_7$-generated complex coefficients and the quaternionic basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_4$. Hence there is another layer of complexity that needs to be factored in in order to correctly understand the natural mass structure that emerges from the exceptional Jordan algebra.

[4]http://brannenworks.com/

so we arrive at the Koide mass formula systematically from first principles. At $(r,\delta) = (\sqrt{2}, \frac{\pi}{6} = \delta_\nu - \delta_{\bar{\nu}})$ we have

$$\frac{\sqrt{m_1 m_2 m_3}}{(\sqrt{m_1}+\sqrt{m_2}+\sqrt{m_3})^3} = \frac{1}{27}. \tag{3.249}$$

Numerically, with ϵ very small and m_p the proton mass:

$$\begin{aligned}
\text{Leptons}: (r,\delta,\mu) &= (\sqrt{2}, \frac{2}{9}+\epsilon, \frac{m_p}{3}) && (3.250)\\
\text{Neutrinos}: (r,\delta,\mu) &= (\sqrt{2}, \frac{2}{9}+\frac{\pi}{12}, \mu_\nu), \quad \mu_\nu = \frac{m_H^2}{m_P} = \sqrt{\frac{2\pi G}{ch}} m_H^2 && (3.251)\\
\text{Anti}-\text{Neutrinos}: (r,\delta,\mu) &= (\sqrt{2}, \frac{2}{9}-\frac{\pi}{12}, \mu_{\bar{\nu}}) && (3.252)\\
(d,s,b)\ \text{Quark Families}: (r,\delta,\mu) &= (\sqrt{3}, \frac{4}{27}, \mu_d) && (3.253)\\
(u,c,t)\ \text{Quark Families}: (r,\delta,\mu) &= (\sqrt{3}, \frac{2}{27}, \mu_u). && (3.254)
\end{aligned}$$

Here the neutrinos are of different mass. Hence they cannot combine into Dirac states and are hence restricted to be charge neutral. The numerical relationships between the quark charge and phases are given by

$$\delta_d = 4Q_d^3, \qquad \delta_u = \frac{1}{4}Q_u^3, \qquad \delta_d = 2\delta_u. \tag{3.255}$$

The existence of a preferred octonionic imaginary unit playing the role of the complex imaginary unit splits up the octonions as a vector space as $\mathbb{O} = \mathbb{C}_{\mathbf{e}_7} \oplus \mathbb{C}^3_{\mathbf{e}_7}$. It then seems natural to suspect that mass has something to do with $\mathbb{C}^3_{\mathbf{e}_7}$ and the set of matrices $\mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7})$ that can be consequently defined. The remaining six terms in the octonionic Dirac operator are therefore vectors $z \in \mathbb{C}^3_{\mathbf{e}_7}$. These can be thought of as the purely imaginary parts of complex quaternions. If z is to be an observable it should be part of $\mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7}) \subset J_3^{\mathbb{O}}$. We can form an element $Z \in \mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7})$ as follows:

$$Z = zz^\dagger = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} (\bar{z}_1\ \bar{z}_2\ \bar{z}_3) = \begin{pmatrix} z_1\bar{z}_1 & z_1\bar{z}_2 & z_1\bar{z}_3 \\ z_2\bar{z}_1 & z_2\bar{z}_2 & z_2\bar{z}_3 \\ z_3\bar{z}_1 & z_3\bar{z}_2 & z_3\bar{z}_3 \end{pmatrix} \in \mathcal{H}(3,\mathbb{C}). \tag{3.256}$$

The physical state space associated with $\mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7})$ is $\mathbb{C}_{\mathbf{e}_7}P^2 = \{X \in \mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7}) : XX = X, \mathrm{tr}(X) = 1\}$. A simple calculation reveals that in order for $Z \in \mathbb{C}_{\mathbf{e}_7}P^2$, the diagonal bits would have to be complex and not real. The diagonal bits are strictly real. Hence $Z \notin \mathbb{C}_{\mathbf{e}_7}P^2$, but $Z \in \mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7})$. This is to be expected since the physical states contain the solutions to the Dirac equation. There is no obvious need for part of the operator to in of itself need to be a physical state. The automorphism group $\mathrm{Aut}(\mathcal{H}(3,\mathbb{C}_{\mathbf{e}_7}))$ is $\mathrm{Isom}_{\mathbb{R}}(\mathbb{C}_{\mathbf{e}_7}P^2) = SU(3)$. The automorphism group itself is invariant under Z_3, the cylcic group of order 3. In order for Z to be a cyclic matrix we require

$$z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 \equiv r \in \mathbb{R}, \quad z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1 \equiv e^{\mathbf{e}_7\phi} \tag{3.257}$$

in which case we obtain

$$Z = \begin{pmatrix} r & e^{\mathbf{e}_7\phi} & e^{-\mathbf{e}_7\phi} \\ e^{-\mathbf{e}_7\phi} & r & e^{\mathbf{e}_7\phi} \\ e^{\mathbf{e}_7\phi} & e^{-\mathbf{e}_7\phi} & r \end{pmatrix}. \tag{3.258}$$

From the theory of circulant matrices applied to the 3×3 case, the eigenvectors are

$$v_0 = \frac{1}{\sqrt{3}}(1, \omega_0, \omega_0^2)^t, \quad v_1 = \frac{1}{\sqrt{3}}(1, \omega_1, \omega_1^2)^t, \quad v_2 = \frac{1}{\sqrt{3}}(1, \omega_2, \omega_2^2)^t \tag{3.259}$$

where $\omega_0 = e^{\frac{\mathbf{e}_7 2\pi 0}{3}}$, $\omega_1 = e^{\frac{\mathbf{e}_7 2\pi}{3}}$ and $\omega_2 = e^{\frac{\mathbf{e}_7 4\pi}{3}}$ are all third roots of unity with $\omega_0 = 1$ in particular. The corresponding eigenvalues are respectively

$$\lambda_0 = r, \quad \lambda_1 = r + e^{\mathbf{e}_7 \phi}\omega_1, \quad \lambda_2 = r + e^{\mathbf{e}_7 \phi}\omega_2 + e^{-\mathbf{e}_7 \phi}\omega_2^2. \tag{3.260}$$

We also have $\det(Z) = \lambda_0\lambda_1\lambda_2$ and in terms of the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.261}$$

we have $Z = rI_3 + e^{-\mathbf{e}_7\phi}P + e^{\mathbf{e}_7\phi}P^2$. Moreover, given two circulant matrices A, B, we have $A + B$ circulant and also AB circulant. Also, $AB = BA$ so the set of all circulant 3×3 matrices form an abelian associative algebra. When restricted to a matrix with complex entries this C^* algebra is isomorphic to the group C^* algebra $\mathbb{Z}/3\mathbb{Z}$. The discrete 3×3 Fourier transform F_3 is given by

$$F_3 = (f_{jk}), \quad f_{jk} = e^{\frac{-\mathbf{e}_7 2\pi jk}{3}}, \quad 0 \leq j, k < 3. \tag{3.262}$$

Explicitly,

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{\mathbf{e}_7 2\pi}{3}} & e^{-\frac{-\mathbf{e}_7 4\pi}{3}} \\ 1 & e^{-\frac{-\mathbf{e}_7 4\pi}{3}} & e^{-\frac{-\mathbf{e}_7 8\pi}{3}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \omega_0 & \omega_1^* & \omega_2^* \\ \omega_0^2 & \omega_1^{*2} & \omega_2^{*2} \end{pmatrix}. \tag{3.263}$$

The three column vectors of F_3 are $\sqrt{3}v_i$, $i = 0, 1, 2$. Thus the matrix $U_3 = \sqrt{3}F_3$ together with $U_3^* = \frac{1}{\sqrt{3}}F_3^{-1}$ diagonalizes Z.

3.9.3 The Clifford braiding theorem

This section is based on results which can be found in [23] and [29]. There exist relationships between certain Clifford algebras and the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We have:

$$C\ell(0,0) \cong \mathbb{R}, \quad C\ell(0,1) \cong \mathbb{C}, \quad C\ell(0,2) \cong \mathbb{H}, \quad C\ell(0,6) \cong \overleftarrow{\mathbb{O}}. \tag{3.264}$$

Since the normed division algebras are isomorphic to certain Clifford algebras, it follows that the normed division algebras themselves contain representations of certain braid groups. The braid groups are the same ones that are relevant to the helon representations of standard model particles that embed into framed loop quantum gravity braided belt networks.

Clifford algebras of the form $C\ell(n, 0)$ contain representations of the circular Artin braid group B_n^c. If we denote the generators of $C\ell(n, 0)$ by $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and the n braid elements by

$\{\sigma_1, \cdots, \sigma_n\}$ then according to the Clifford Braiding Theorem we can represent the braid elements in terms of the Clifford elements by

$$\sigma_k = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{k+1}\mathbf{e}_k), \quad 1 \leq k < n, \tag{3.265}$$

$$\sigma_n = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_1\mathbf{e}_n). \tag{3.266}$$

Since the Clifford elements used in the Clifford Braiding Theorem belong to the even part $C\ell^+(n,0)$ only, and since $C\ell^+(n,0) \cong C\ell(0,n-1)$, it follows that we can also represent the braid group by elements in $C\ell(0,n)$. So for the complex, quaternionic and (chained) octonionic normed division algebras we have

$$\mathbb{C} \cong C\ell(0,1) \cong C\ell^+(2,0) \longrightarrow B_2, \tag{3.267}$$

$$\mathbb{H} \cong C\ell(0,2) \cong C\ell^+(3,0) \longrightarrow B_3^c, \tag{3.268}$$

$$\overleftarrow{\mathbb{O}} \cong C\ell(0,6) \cong C\ell^+(7,0) \longrightarrow B_7^c. \tag{3.269}$$

Representations of B_2 in terms of $C\ell(0,1)$, $C\ell^+(2,0)$ and $\mathbb{C}$

For $C\ell(0,1)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1 - \mathbf{e}_1)$ and $\sigma_1^{-1} = \bar{\sigma}_1$.
For $C\ell^+(2,0)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_2\mathbf{e}_1)$, $\sigma_1^{-1} = \frac{1}{\sqrt{2}}(1-\mathbf{e}_2\mathbf{e}_1) = \hat{\sigma}_1$ where $\hat{\sigma}_1$ is the Clifford reverse of σ_1.
For $\mathbb{C}$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1+i)$, $\sigma_1^{-1} = \bar{\sigma}_1 = \frac{1}{\sqrt{2}}(1-i)$.
In $\mathbb{C}$ the order of the braid generator is eight so not every braid can be represented in $\mathbb{C}$. But it turns out that the relevant ones can.

Representations of B_3^c in terms of $C\ell(0,2)$, $C\ell^+(3,0)$ and $\mathbb{H}$

For $C\ell(0,2)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{21})$, $\sigma_2 = \frac{1}{\sqrt{2}}(1 - \mathbf{e}_2)$ and $\sigma_3 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_1)$.
For $C\ell^+(3,0)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{21})$, $\sigma_2 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{32})$ and $\sigma_3 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{13})$.
For $\mathbb{H}_{\mathbf{e}_7,i}$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_7)$, $\sigma_2 = \frac{1}{\sqrt{2}}(1+i)$ and $\sigma_3 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_7 i)$.

Representations of B_7^c in terms of $C\ell(0,6)$, $C\ell^+(7,0)$ and $\overleftarrow{\mathbb{O}}$

For $C\ell(0,6)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{21})$, $\sigma_2 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{32})$ and $\sigma_3 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{43})$, $\sigma_4 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{54})$, $\sigma_5 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{65})$ and $\sigma_6 = \frac{1}{\sqrt{2}}(1 - \mathbf{e}_6)$, $\sigma_7 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_1)$.
For $C\ell^+(7,0)$ we have $\sigma_1 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{21})$, $\sigma_2 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{32})$ and $\sigma_3 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{43})$, $\sigma_4 = \frac{1}{\sqrt{2}}(1+\mathbf{e}_{54})$, $\sigma_5 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{65})$ and $\sigma_6 = \frac{1}{\sqrt{2}}(1 - \mathbf{e}_{76})$, $\sigma_7 = \frac{1}{\sqrt{2}}(1 + \mathbf{e}_{17})$.
For $\overleftarrow{\mathbb{O}}_{l,i,j}$ we have $\sigma_i = \frac{1}{\sqrt{2}}(1 + \overleftarrow{\mathbf{e}_{i+1}\mathbf{e}_i})$ and $\sigma_7 = \frac{1}{\sqrt{2}}(1 + \overleftarrow{\mathbf{e}_1\mathbf{e}_7})$. Here $\mathbf{e}_1 = l$, $\mathbf{e}_2 = i$, $\mathbf{e}_3 = li$, $\mathbf{e}_4 = lj$, $\mathbf{e}_5 = ij$, $\mathbf{e}_6 = lij$ and $\mathbf{e}_7 = j$.
In all cases, inverses are given in $C\ell(0,n-1)$ by Clifford conjugation, inverses in $C\ell^+(n,0)$ are given by Clifford reversion and inverses in $\mathbb{K}$ is given by Cayley-Dickson conjugation.

3.9.4 The non-observability of individual quark states

We can conceivably relate the fermionic braided helons to observables by taking a trace. The trace closure[5] of any of the elementary fermionic helons gives the unknot. For quarks, once this is done, the colour is lost because one cannot point to part of an unknot and say "that is where the twisting is and that belongs to this particular strand." The trace closure of a braid therefore results in a form of degeneracy in the color quantum numbers. But quarks have a well-defined colour so this contradiction between that fact, and the lack of observability of color forbids an individual quark from being observable. Hence the helon model provides a natural topological explanation for the fact that individual quark states have not been observed in nature.

3.9.5 Gauge and particle symmetries from $\mathbf{Aut}(J_3^{\mathbb{O}}) = F_4$

This section mentions some results found in [69]. Recall that

$$\begin{aligned} J_3^{\mathbb{O}} &\equiv \mathcal{H}(3,\mathbb{O}) = \{X \in \mathrm{Mat}(3,\mathbb{O} : X^\dagger = X)\}, \quad X^\dagger \equiv \bar{X}^t, \\ \mathbb{O}P^2 &\equiv \{X \in J_3^{\mathbb{O}} : X * X = 0, \quad \mathrm{tr}X = 1\} = \{X \in J_3^{\mathbb{O}} : X \circ X = X, \quad \mathrm{tr}X = 1\}\}. \end{aligned}$$

The automorphism group of the exceptional Jordan algebra is the exceptional Lie group F_4. We have

$$\begin{aligned} \mathrm{Aut}(J_3^{\mathbb{O}}) &= F_4 \equiv \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(J_3^{\mathbb{O}}) : \alpha(X \circ Y) = \alpha X \circ \alpha Y\} = SU(3,\mathbb{O}), \\ \mathrm{tr}(\alpha X) &= \mathrm{tr}(X), \quad \det(\alpha X) = \det X, \quad (\alpha^{-1})^t (X * Y) = \alpha X * \alpha Y. \end{aligned} \tag{3.270}$$

The physical states corresponding to elements of the state space $\mathbb{O}P^2$ of $J_3^{\mathbb{O}}$ must be compatible with the automorphism group F_4 in some sense. This Lie group has rank 4 but it also has three subgroups which also have rank 4 so it has three maximal rank subgroups. They are

1. $Spin(9)$,
2. $(SU(3) \times SU(3))/Z_3, \quad Z_3 = \{(E, E), (w_1 E, w1E), (w_1^2 E, w_1^2 E)\}, \quad w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{e}_1,$
3. $(Sp(1) \times Sp(3))/Z_2, \quad Z_2 = \{(1, E), (-1, -E)\}.$

These maximal rank subgroups do not form a subgroup chain. Thus there must be some gauge redundancy in the intersection of these maximal rank subgroups. However, in addition to this we need to preserve the $SU(3)_c$ maximally totally isotropic subgroup so this rules out consideration of $Sp(1) \times Sp(3)/Z_2$. We see then, that

$$Spin(9) \cap (SU(3) \times SU(3))/Z_3 = S(U(2) \times U(3)). \tag{3.271}$$

But

$$S(U(2) \times U(3)) = SU(3) \times SU(2) \times U(1)/Z_6 \tag{3.272}$$

which we recognize mathematically as precisely the gauge group of the standard model.

[5] See [76] for more on the trace closure of a braid.

3.10 Quantum gravity

Before concluding this section, we make some remarks about how the paradigm developed here should also give quantum gravity. Firstly, we give an extremely concise description of the core parts of covariant loop quantum gravity, based on Rovelli [11]. In covariant loop quantum gravity one has a triangulation Δ, its boundary $\partial\Delta$, the 2-complex Δ^* dual to the triangulation, and the boundary graph $\Gamma = (\partial\Delta^*) = (\partial\Delta)^*$. The state space is based on the infinite-dimensional $SL(2,\mathbb{C})$ (p,k)-representations

$$V^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j \tag{3.273}$$

where $p > 0$ and k is a non-negative half-integer. The j labels the the $SU(2) \subset SL(2,\mathbb{C})$ representations in terms of spin-j. The linear simplicity constraint restricts the quantum numbers to $(p,k) = (\gamma j, j)$ which puts them into one-to-one correspondence with the $SU(2)$ representations. The spin-network states are given by $|\gamma j, j; j, m\rangle$, or in terms of wavefunctions, by $\psi(g) = \sum_{jmn} D^{(\gamma j,j)}_{jm,jn}(g)$. The further restriction to the physical state space of gauge invariant functions is

$$L_2[SU(2)^L/SU(2)^N]_\Gamma = \oplus_{j_\ell} \otimes_n \mathrm{Inv}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \tag{3.274}$$

where L is the number of links ℓ, and N is the number of nodes n, of the graph Γ. The quantum dynamics is described by the transition amplitude

$$W_\Delta(U_\ell) = \langle W_\Delta | U_\ell \rangle = \mathcal{N} \int dU_e \int dL_f e^{\frac{i}{8\pi\hbar G} \sum_f \mathrm{Tr}(U_f L_f)} \tag{3.275}$$

where $U_f \in SL(2,\mathbb{C})$ are configuration space variables and $L_f \in sl(2,\mathbb{C})$ are the canonically conjugate momentum space variables. More descriptively, $U_f = U_{e_1} \cdots U_{e_n}$ where f denotes the face which is bounded by n edges $e_1, \cdots e_n$ on the two-complex. The problem with this theory as here presented is that the transition amplitude has infrared divergence. The usual solution is to q-deform to the coquasitriangular Hopf algebra $SL_q(2,\mathbb{C})$ and the quasitriangular Hopf algebra $\mathcal{U}_q(sl(2,\mathbb{C}))$. This has the effect of thickening up the spin networks (i.e. framing them), generalizing the three-dimensional spin-networks to braided belt networks and generalizing the four-dimensional spin foam to braided belt foam. The helons naturally embed into these braided belt networks contained in the braided belt foam [8, 6, 7].

Now consider the fermionic mass states, which, as we have seen braid one another under B_3^c. If they braid one another then the fermions live in the braided category of objects which are respected by the quantum Lorentz group $U_q(sl(2,\mathbb{C}))$. The quantum spacetime states must belong to the algebra of symmetries of this, which is the coquasitriangular Hopf algebra $SL_q(2,\mathbb{C})$. Hence the quantum gravity spacetime configuration space should be $SL_q(2,\mathbb{C})$ and the quantum gravity spacetime momentum space should be $U_q(sl(2,\mathbb{C}))$, but as we have just seen, this gives loop quantum gravity, that is, so long as the linear simplicity constraint also emerges. To see that the linear simplicity constraint emerges, note that[56, p.353]

$$U_q(sl(2,\mathbb{C})) \cong BU_q(su(2)) \bowtie U_q(su(2)) \tag{3.276}$$

so the quantum gravity spacetime spin foam states of $SL_q(2,\mathbb{C})$ must be in one-to-one-correspondence with the spin-network states of $SU_q(2)$. Thus the classical limit of the theory is general relativity, with positive cosmological constant. So loop quantum gravity naturally emerges from the theory of everything, tracing back to the constraint that the fermionic octonionic vector space decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ is preserved. The total configuration space is thus obtained by taking braided tensor products of copies of $U_q(sl(2,\mathbb{C}))$ where the braiding is defined by the quantum Lorentz R-matrix, which in turn, comes from the quasitriangular structure $R = (\rho \otimes \rho)\mathcal{R}$ which defines $U_q(sl_2)$.

3.11 Physics: Conclusions and open questions

We have started with a very general set of starting axioms:

1. The theory should be quantum,
2. The fundamental symmetry principle is that there does not exist preferred bases for a state in the state space,
3. The configuration space consists of the single inseparable self-interacting universe.

which emerge as necessary for harmony with the Bible. The resulting Hilbert space topology gave rise to a bundle structure giving rise to projective lines over the Cayley-Dickson algebras of which the largest normed division algebra is the octonions. The only higher projective geometry that projects down to copies of the octonionic projective line is the octonionic projective plane.[6] Finding specific $P \in \mathbb{O}P^2$ amounts to solving the equation $P * P = 0$. This gives rise to what is mathematically equivalent to a 1+9-dimensional massless momentum space Dirac-type equation.

We showed that the overall 0+6-dimensional mass term is not the physical mass if all elements are nonzero. Projections are needed which takes the mass term to a physical mass belonging to one of the three generations of fermionic solutions to the Dirac-type equation. Moreover, these projections generate the circular braid group B_3^c. We showed that three generations of helons naturally emerge from this structure. We were thus able to give a natural three generation extension to the one generation helon model. Moreover the existence of a projection map down to the permutation group allows for the representation of helons in terms of 3×3 matrices. The leptons are given in terms of circulant matrices while the quark matrices are not. The Koide mass formulas thus emerge from this structure and so they should no longer be considered as mere interesting coincidence formulas. Furthermore, we also showed how the PMNS and CKM mixing matrices emerge, although we did not look into the numerical values of the matrix elements.

The three generations of helons were linked to the one generation Furey model of fermions being minimal left ideals of the complex chained octonion algebra[22, 21], thereby extending the one generation results of Gresnigt in [23]. A slightly different point of view is taken by Stoica[68] who also uses $\mathbb{C}\ell(6)$ to get a generation of fermionic states, complete with Lorentzian and weak quantum numbers.[7] Moreover the standard model gauge group is exactly the intersection of the $SU(3)$-preserving maximal rank subgroups of $\mathrm{Aut}(J_3^{\mathbb{O}}) = F_4 = SU(3,\mathbb{O})$ See the appendix based on [69] for details.

[6] We did not consider trace 2 projections of $J_3^{\mathbb{O}}$ because they are in one-to-one correspondence with the trace 1 projections by virtue of the self-duality of $\mathbb{O}P^2$ whereby points are mapped to lines and lines are mapped to points[5].

[7] and also a zeroth order prediction of the Weinberg angle.

As shown in the quantum gravity section, framed covariant loop quantum gravity naturally emerges, which, in the classical limit reproduces general relativity (i.e. classical gravity). The emergence of quantum gravity is fundamentally a natural result of preserving the fermionic octonionic vector space decomposition $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$. The number of dimensions of space and time also naturally emerge uniquely.

We finish with the following open questions, which form natural lines of needed future investigation in order to further scrutinize our presented paradigm:

1. What exactly is the physical effect of the existence of zero divisors that come up in the higher Cayley-Dickson algebras in this paradigm where we view the Universe as an inseparable self-interacting whole? Zero divisors of Cayley-Dickson algebras are still somewhat mysterious. However progress has been made towards understanding them. See [62, 15, 61, 63].

2. Can an exact *Koide* style formula or perhaps a very different looking mass formula for the non-circulant quarks be derived?

3. Given that the PMNS and CKM matrices arise within this exceptional Jordan algebra structure, can a prediction be made about the numerical values of their matrix elements?

4. How can the three generation helon model presented here be formulated explicitly in terms of the circular braid group B_6^c that is generated by $\mathbb{C}\ell(6)$ via the Clifford braiding theorem of [29]?

Chapter 4

The Existence Of God; Reversing The Argument

4.1 Proof of the existence of God, up to an additional human consciousness axiom

We have noticed that if one starts off with the relationship between God and His Son described in the Bible, we get precisely the particles and interactions and spacetime that we see around us. Moreover, we get it uniquely.

The argument is reversible.

Given the standard model particles and forces we can run the lines of deductions and arrows of implications the other way and conclude that there exists a Creative Duo with precisely the relationship that the Bible describes as being held by God and His Son. The only thing that might not necessarily be a given is that the "God" and the "Son" are conscious. We will now comment on consciousness. It is empirically/experimentally known that consciousness does not originate from the physical brain. No physical origin of consciousness is known. Here are some basic physical properties of consciousness:
1. Consciousness can be translated. Someone possessing the property of consciousness can walk along the street and still be conscious after the translation along the street.
2. Consciousness can be created by reproductive procedures from entities which themselves are conscious. In this sense, consciousness can be begotten from consciousness. In other words, consciousness can beget consciousness under the right circumstances.
3. There are no known cases of consciousness coming into existence by any other means other than being begotten from previously existing consciousness (remember that consciousness does not originate from the physical material brain).

So then, if we accept:
1. the known standard model particles and their forces,

2. the known physical properties of spacetime,
3. there exists consciousness in the obvious sense of being genuinely self-aware possessing genuine free will,

then reversing the above arguments together with tracing consciousness back to the first two states reveals that the first two states are themselves conscious, in which case we may conclude/identify:

$$|s\rangle_G \leftrightarrow \mathbf{God}, \qquad |\psi_s\rangle_S \leftrightarrow \mathbf{God's \quad Son}. \tag{4.1}$$

So we have a sort of "proof" up to consciousness axioms for the existence of both the Biblical God and His Son.

4.2 Deducing God's Law from our knowledge of particles and spacetime

From the physics we see from a mathematical stand point that there does not exist a preferred basis at any level of iteration. Hence there is reciprocity between "neighboring states." If we recognize love as being the fundamental property of serving an entity that is distinct from itself then the mathematical structure at all levels reveals the commandment:

Thou shalt love thy neighbor as thyself.

Moreover, the end of all parts of all interactions is God. Hence the other basic Law contained consistently throughout the entire mathematical structure is:

Thou shalt love The Lord thy God with everything.

These two Laws are now clearly seen to form the basis of all physical natural law. Furthermore, given the fact that consciousness has been embedded into the structure, in order to preserve the structure, the consciousness addition to the structure must also harmonize with the same physical laws in order to preserve the physical structure. Hence the two great commandments are also moral, applying also to conscious intelligent entities.

By its very nature we now see clearly that God's Law is immutable. God cannot alter or do away with His Law. It is impossible to both change His Law and maintain the existence of creation.

We now consider: **sin**, which is defined in 1 John 3:4 as being the transgression of God's Law. Before we look at sin, we summarize the broad results so far:

Given the empirical observations about quarks, and given the empirical observations about consciousness, one can **deduce/conclude** that:

1. God exists,

2. God has a Son,

3. God executes His will through His Son,

4. God's Son does nothing that is outside of God's will,

5. God has a two-fold great Law consisting of both "love God with everything" and "love thy neighbor as thyself,"

6. This two-fold great Law is universal, dictating to both the intelligent conscious entities and also to the physical nature in all aspects at all scales from the subatomic scale right up to the largest of cosmological scales.

7. The Bible's claims concerning God, His Son, Their relationship, God's Law, how to get knowledge from the Foundation of all knowledge, is absolutely correct, as one would expect if the Bible is to be recognized as the infallible totally inspired Word of God.

4.3 The physics of sin

1 John 3:4. Whosoever committeth sin transgresseth also the law: for sin is the transgression of the law.

We have seen that God's Law is imprinted all through the structure of creation and under girds all natural physical relationships. Love God with everything is seen in nature by all existence stemming from God, the Source of both the initial creation and also the continual upholding of creation. We have also seen how the love thy neighbor as thyself Law is encoded at all levels of the mathematical structure in terms of the absence of a preferred coordinate basis and through the necessary Observer-State Symmetry Principle. All horizontal arrows of implication are reciprocal. Sin is the transgression of the Law so sin must be the breaking of these arrows of implication. Love and sin are opposites. Love basically means giving while sin basically means taking. Note that taking is not the same thing as receiving. Taking implies taking that which is not naturally coming ones way, sent by a giving entity.

We also note that purely physical entities with no consciousness are unable to decide to break God's Law so cannot be sources of sin. Thus in order to study the effects of sin on creation we need to consider those entities that are both conscious and tied to creation. We note at this point that angels are left out of the discussion.

Let us start our consideration of the effect of sin on creation by seeing hypothetically what would happen if either God or His Son were to sin. If either or both were to become "takers" rather than "givers" then the relationship They have which is crucial for physics as we know it would be lost. The quantum state that is the universe would be instantly nonlinearly mapped from an entangled quantum state to an unentangled quantum state. There is no interactive structure in the unentangled situation which is so critical in giving rise to spacetime and fermionic matter with all of the known forces etc. It would all simply cease to exist since the upholding Agency sustaining the structure has become unentangled.

The fact that the universe still exists is proof that neither God, nor His Son have ever sinned. From this we can therefore conclude that They have never bore false witness. It follows then that every word of God can be trusted completely. Concerning promises God has made in advance, He must have every intent to fulfill all of His promises. Otherwise He would have bore false witness and in doing so blotted the universe from existence.

The next hypothetical source of sin would be from the first **created** conscious entities. Let us now focus on Adam. Dominion over creation was given to him. A big difference between Adam and The Father and Son Duo however is that unlike The Father and The Son, Adam does not continually sustain and uphold the existence of creation. Hence if Adam were to sin, it does not pose an immediate unavoidable existential threat to the universe (and of course, we know that Adam has sinned). God is a God of order. Since God gave Adam dominion over the Earth, it follows that there exist (before sin anyway) arrows of implication from God over the earth which flow through Adam. When Adam sinned, he reversed some arrows of implication. There were direct existential arrows of implication from God directly to every particle under Adam's dominion but also arrows of implication that pass through Adam. The latter arrows of implication were reversed, hence a localized disentanglement presumably occurred. Adam's consciousness had control over his local region within the universe. Presumably this stopped the flows of action that were supposed to be taking place from God through Adam to his dominion. Hence some kind of functionality must have ceased. Since everything is tied together through the creation the improper functioning must have spread. Logically the universe as a whole is simply not functioning how it should. Hence we would expect that the universe is in terminal decline. Indeed this is the case. Consider Isaiah 51:6 and Hebrews 1:10-12,

Isaiah 51:6. Lift up your eyes to the heavens, and look upon the earth beneath: for the heavens shall vanish away like smoke, and the earth shall wax old like a garment, and they that dwell therein shall die in like manner: but my salvation shall be for ever, and my righteousness shall not be abolished.
Hebrews 1:10. And, Thou, Lord, in the beginning hast laid the foundation of the earth; and the heavens are the works of thine hands:
Hebrews 1:11. They shall perish; but thou remainest; and they all shall wax old as doth a garment;
Hebrews 1:12. And as a vesture shalt thou fold them up, and they shall be changed: but thou art the same, and thy years shall not fail.

One sin physically destroys the entire universe. Eventually. This supports yet another Bible doctrine, namely that the Law of God is permanent and unchangeable. We now move on from mathematical physics to cosmology, and show the perfect harmony between cosmology as commonly understood by cosmologists, and the Bible. In fact, the standard cosmology (or its fractal bubble time-scape cosmological correction) provides totally independent strong evidence that the Bible is indeed trustworthy as the true and infallible Word of God. We assume the reader is broadly aware of the standard model of cosmology.

Chapter 5

The Universe: Cosmology And Creation

In previous chapters we looked at the relationship between the Bible and physics. We saw the physics theory of everything emerge from the Bible. Up to conscousness axioms we have then looked at everything concerning physics. What we have not done yet however is look at the universe on the large scale; i.e. we have not had anything to say about cosmology. In this chapter we take the standard model of cosmology and do some additional analysis motivated by the fact that the nonzero expansion rate induced time dilation. The age of the universe in the standard model of cosmology is the proper time, i.e. the time experienced by somebody, if someone had have been in the universe and went along for the ride and got to watch everything from the point of view of being an internal observer. This stretching/dilated clock time for the age of the universe in the standard model is around $13.8Gyr$ (i.e. 13.8 billion years). The standard model of cosmology assumed perfect isotropy and homogeneity at all times. A more realistic picture however, acknowledges that in terms of the stretching clock time, homogeneity started becoming less and less valid from about $10Gyr$ ago. The real universe in which we live is composed of large voids, bounded by walls/filaments of galaxy clusters. This inhomogeneity has induced a “timescape” whereby, due to gravitational time dilation, clocks tick at different rates at different positions in the universe. A clock at the centre of a large void might register a universe age several billion years older than a clock in an average filement location such as our own location in a wall/filament of galaxy clusters. Taking all of this into account gives an earth-based age of the universe time of around $15Gyr$. The standard model of cosmology assumes via homgeneity that the FRW time parameter is universal as a clock time at all locations in the universe. In reality, this is only an averaged time and must be careful about how to related their local clocks to this averaged FRW clock. In the standard model, the failure to take into account the nontrivial nature of the cosmological timescape has led to the incorrect conclusions that the data implies that the expansion of the universe has been accelerating and that there hence must exist dark energy. So there is no dark energy in reality.

One might wonder what all of this has to do with the Bible. The Bible says God created everything in six days with literal language, strongly implying that the intent of the Bible is to say that in terms of our clocks, the universe was created in six 24 hour days. Many Christians have dismissed the standard model of cosmology and any improvements based on the fact that six days does not sound like billions of years. Many atheists have dismissed the Bible based on the fact that six days does not sound like billions of years.

The big thing that both the Christians and the Atheists have neglected to take into account is that the nonzero expansion rate of the universe had the physical effect on time that the rate of time flow was increased relative to the rate of time flow of a non-expanding universe.

Atheists have ignored this because this has no bearing on increasing ones levels of science knowledge. Working out different clock times does not constitute new physics so researchers do not bother to think about it.

Christians have ignored this, probably because it did not occur to them to think about it.

Dr. Gerald Schroder has thought about this however, and has shown that if one takes the current standard model age of the universe, corrected for its actual inhomogeneities (in practice by using a slightly increased age time of 14 Gr to offset the apparent expansion acceleration effect), and one calculates what the universe age time is in terms of non-expanding clocks, one gets six days! It should be emphasized that this was done without any recourse to the Bible, and without any violence being done to the standard model of cosmology. The fact that the non-expanding clock time age of the universe turns out to be six days is a powerful independent verification that the Bibllical creation time is correct.

The Christian and/or Atheist might then skeptically point out that this verification of the Bible breaks down when we transition from the sixth day of creation to the seventh day. After all, Biblically speaking the Universe is about 6000 years old which is very different to both six days old and 14 billion years old. Dr. Gerald Schroeder assumes the Universe is still expanding which preserves this problem. In this case the preservation of the Bible as historically accurate rests on saying that after the creation of man, God starts talking in terms of a different clock that is convenient for Adam and so fom day 7 on, our usual every day clocks are used, whereas for creation, pre-Adam, God spoke in terms of the original operational light-based clock from when the universe had a temperature of $10.9 \times 10^{12} K$. Here, we differ as follows:

1. Dr. Gerald Schroeder takes the view that the Universe **is still** expanding. This means that our everyday clocks are the same as the clock rates of matter observers at any time during the rest of the expansion history. Hence according to these same every day clocks of ours, the Universe is 14 billion years old plus 6000 years which again is about 14 billion years old. If the universe is indeed still expanding then this view would be right. God would be speaking about two different clocks in Genesis 1 and 2, which are related in a well-defined way by General Relativity.

2. We take the view that the Universe is **not still** expanding, but that God stopped expanding the Universe at the end of His creative work, which was completed at the end of the sixth day. The first six days of clock time were stretched out to around 14 billion years by General Relativity due to the fact that the expansion rate was nonzero. When God stopped expanding the Universe, the expansion rate became zero so the clocks became non-stretched so day 7 and all days since have been as we naturally think of time today. If one were to somehow be able to view the first six days through the lenz of their unstretched clock time, one would see 14 billion years worth of activity take place such that their watch only ticked off 6 days. It would be like watching a movie on fast forward, so fast that one couldn't keep up.

The abruptness of God ceasing to expand the Universe causes no physics problems because we are talking about the entire 1+3-dimensional spacetime manifold. Not just some three-dimensional spatial hypersurface. Also, the constants of nature remained constant, as measured by any matter-based observers inside the universe during the six days of expansion. To a hypothetical outside test observer with a zero-expansion rate watch on his wrist it would have appeared as though for the first six days the speed of light c was slowing down and plank's constant h was increasing (however the fine structure constant α and the product hc would have still appeared constant). But an actual matter-observer in the universe would never notice because internally, this would be gauge freedom and so the physics is unaffected. So no change in light speed or Planks constant occurred as far as anything that is part of the Universe would be concerned.

In terms of our current everyday non-expanding clock rate time, the universe is about 6000 years old. According to the expanding clock rate time that was operationally useful during the first 6 days, the Universe would be around 14 billion years old (actually, older once the corrections due to structure formation induced inhomogeneities have been taken into account)

One might worry that if the universe stopped expanding 6000 years ago this should have been noticed. We note that at this point that expansion is only seen on scales larger than our local gravitationally bound local group of galaxies which is about 10 million light years across. So non-redshifted light, which has only been travelling for about 6000 years has not had anywhere enough time to reach us for observation. Hence there is no scientific evidence for either current expansion or current non-expansion. The 14 billion years of expansion time according to proper time matter clocks corresponds to 6 days in zero-expansion-rate time independently of the Bible which is strong evidence for the Bible. The Bible also says at least 11 times that God stretched out the heavens. So the expansion of the Universe is an important Bible doctrine we are obviously supposed to take careful note of. Also importantly, is that the phrasing and context is always past tense for creation week. Given the flawless reliability of all of the testable elements of the Bible, it follows that we should take the view, not in anyway contradicted by observation, that the expansion stopped at the completion of creation. General Relativity then makes it a totally natural and smooth transition from the 6th day to the 7th day and onwards.

In the rest of this chapter we wish to:

1. Present Dr. Gerald Schroeder's calculation of the first light-clock age of the universe.

2. Derive Dr. Gerald Schroeder's calculation from General Relativity. The derivation assures us that the equation is valid.

3. See what the Bible says about the expansion and creation of the universe. We also look at Bible prophecy that is related to the Bible denying views that people would develop in the time leading up to Jesus second coming.

5.1 Schroeder's calculation

Schroeder's calculation, [67], starts off with the first light-clock time t_{cr} and the exponential equation

$$A(t_{cr}) = A_0 e^{-\frac{\lambda t_{cr}}{1\text{ day}}} \tag{5.1}$$

where $\lambda = \ln 2 = 0.693$, i.e. one time constant. The age of the universe in terms of current clocks t_{st} ("st" is here short for "stretching" time) is taken as the integral of this function from $t_{cr} = 0$ to $t_{cr} = 5.5$ days. The initial value A_0 is said to be the instantaneous ratio of the threshold energy of a proton to the current energy of space (as temperature). Specifically,

$$A_0 = \frac{10.9 \times 10^{12} K}{3.03K} = 3.59736 \times 10^{12}. \tag{5.2}$$

The temperature $T = 3.03K$ is the 10% corrected CMB temperature to offset the apparent affect of the acceleration of the expansion of the universe. It is only within the standard model FRW paradigm that the data implies an accelerating expansion rate. If one takes into account the real inhomogeneous observations that we are by volume in a void dominated universe where an FRW-averaged positon is located somewhere in an average void, and not in a wall of galaxy clusters such as where we are, then one realises that our wall time is not the same as the FRW clock time. Taking this into account leads to the conclusion that the expansion rate has no acceleration and that there is instead an apparent acceleration effect due to the nontrivial **timescape** induced by gravitational time dilation which is present by virtue of having this void-wall type structure to the universe. For more on these matters which might not be familar to the reader, consult the many papers developing this subject of timescape cosmology, starting with [73, 12, 41, 42, 36, 43, 71, 44, 45, 46, 47, 48, 39][72, 49, 64, 35, 50, 38, 37, 31, 40, 32, 30, 33, 34].

One gets:

$$\begin{aligned} t_{st} &= \int_0^{5.5} A_0 e^{-\frac{\lambda t_{cr}}{1\,\mathrm{day}}} dt_{cr} = -\frac{3.59736 \times 10^{12}}{0.693}\, e^{-0.693 t_{cr}}\Big|_0^{5.5} = -5.19099529 \times 10^{12} \left(e^{-3.8115} - 1\right) \\ &= 0.9779 \times 5.19099529 \times 10^{12} = 5.076 \times 10^{12}\,\mathrm{days} = 13.9 \times 10^{9}\ \mathrm{yr}. \end{aligned} \tag{5.3}$$

If we were to instead integrate from 0 to 6 we would get:

$$-5.19099529 \times 10^{12} \left(e^{-4.158} - 1\right) = 5.1098141 \times 10^{12}\,\mathrm{days} = 13.99 \times 10^{9}\,\mathrm{yr}. \tag{5.4}$$

Day one: Integrate from 0 to 1:

$$-5.19099529 \times 10^{12} \left(e^{-0.693} - 1\right) = 2.5951144 \times 10^{12}\,\mathrm{days} = 7.10523 \times 10^{9}\,\mathrm{yr}. \tag{5.5}$$

Day two: Integrate from 1 to 2:

$$-5.19099529 \times 10^{12} \left(e^{-1.386} - e^{-0.693}\right) = 1.2977488 \times 10^{12}\,\mathrm{days} = 3.55313985 \times 10^{9}\,\mathrm{yr}. \tag{5.6}$$

Day three: Integrate from 2 to 3:

$$-5.19099529 \times 10^{12} \left(e^{-2.079} - e^{-1.386}\right) = 0.64897 \times 10^{12}\,\mathrm{days} = 1.77683224 \times 10^{9}\,\mathrm{yr}. \tag{5.7}$$

Day four: Integrate from 3 to 4:

$$-5.19099529 \times 10^{12} \left(e^{-2.772} - e^{-2.079}\right) = 0.324533 \times 10^{12}\,\mathrm{days} = 0.888547303 \times 10^{9}\,\mathrm{yr}. \tag{5.8}$$

Day five: Integrate from 4 to 5:

$$-5.19099529 \times 10^{12} \left(e^{-3.465} - e^{-2.772}\right) = 0.16229 \times 10^{12}\,\mathrm{days} = 0.4443 \times 10^{9}\,\mathrm{yr}. \tag{5.9}$$

Day six: Integrate from 5 to 5.5:

$$-5.19099529 \times 10^{12} \left(e^{-3.8115} - e^{-3.465}\right) = 0.04754 \times 10^{12}\,\text{days} = 0.13015919 \times 10^{9}\,\text{yr}. \tag{5.10}$$

If we integrate from 5 to 6:

$$-5.19099529 \times 10^{12} \left(e^{-4.158} - e^{-3.465}\right) = 0.08116 \times 10^{12}\,\text{days} = 0.2222 \times 10^{9}\,\text{yr}. \tag{5.11}$$

Hence in terms of expanding clock time:

Day 6: From 6000 (unexpanded years) to 222.2 million years ago.
Day 5: From 222.2 million years ago to 666.5 million years ago.
Day 4: From 666.5 million years ago to 1.555 billion years ago.
Day 3: From 1.555 billion years ago to 3.3318 billion years ago.
Day 2: From 3.3318 billion years ago to 6.88497 billion years ago.
Day 1: From 6.99497 billion years ago to 13.99 billion years ago.

5.2 Is the creation time calculation physically well-defined?

We have no idea how Dr. Schroeder originally wrote down his equation. In this section we take a more systematic approach to see if we can actually derive the exponential equation used in the previous section. If we are successful, we should be able to more properly understand any assumptions and other circumstances surrounding the validity of the creation time calculation. We will consider mostly classical physics with an intended domain of validity containing at least the history of the universe from now going back to when the first light clocks physically became operationally well-defined.

The universe is a single inseparable entangled quantum state. All observers are internal and all measurements are internal and relative. A notion of absolute size of the classically emergent four-dimensional spacetime manifold does not affect any physics since size is operationally determined by internal observers and all measurements of distance are relative. This observation, together with the observation that the universe is expanding, results in the conclusion that we should regard the entire four-dimensional manifold as expanding and not just a three-dimensional spatial hypersurface. In this sense the spacetime geometry should be conformally invariant. The domain of validity of this set-up remains well-defined as long as the physically well-defined light-based ways of measuring space and time hold. This constraint on the validity of our outlook places the first operational time at the moment when the universe cooled enough to reach the threshold energy of proton-antiproton pair creation, which is known from experiment to be at $10.9 \times 10^{12} K$. From that moment forward, we regard the universe as physically the same at all points in both space and time. Obviously this is not perfectly true. For instance, although the universe appears to be isotropic, and by assumption, also homogeneous for the first 4-5 billion years, structure formation increasingly deviated from the Hubble flow since then. The induced gravitational energy gradients have since created no-uniformity across space for the flow of time with about a 38% variation. If one is happy worrying about such higher order corrections until later, then the universal assumption of spacetime scale invariance is the obvious zeroth order assumption to make. Any corrections to zeroth order results should not be expected to be more than around 38%.

In FRW cosmology the scale factor $a(t)$ is in general not able to be found analytically, except in special cases where in various epochs different contributions to the density is ignored. Apart from the cosmological epoch of the last roughly 4-5 billion years, the solutions for the scale factor are various power laws. The current cosmological epoch exists under the circumstances where the isotropy conditions have been most broken so is outside of the zeroth order domain of validity of our expanding spacetime scale invariance ansatz. Hence we consider the metric

$$ds^2 = dy^2 - Y^{2f}(dx^2 + dy^2 + dz^2) \tag{5.12}$$

for constants Y and f. Now we perform the following change of variables:

$$y \to at, \quad x \to a^{1-f}x, \quad y \to a^{1-f}y, \quad z \to a^{1-f}z. \tag{5.13}$$

Since in principle Y can be rescaled, we can set $f = 1$. We then have

$$ds^2 = a^2(t)(dt^2 - dx^2 - dy^2 - dz^2) = a^2(t)(dt^2 - dr^2 - r^2 d\Omega^2), \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{5.14}$$

We are interested in being able to say something about creation time (i.e. the non-expanding instantaneous first light-clock time), which can be thought of as belonging to a fixed non-expanding coordinate system which coincides with the expanding coordinate system at $t = 0$. The metric has the form

$$g_{\mu\nu}(t, \mathbf{x}) = h(t)g'_{\mu\nu}(t, \mathbf{x}). \tag{5.15}$$

The metric $g_{\mu\nu}(t + t_0, \mathbf{x})$ evaluated at $(t + t_0, \mathbf{x})$ is equivalent to the metric $g_{\mu\nu}(t, \mathbf{x})$ evaluated at $(t, \mathbf{x})$ if

$$g_{\mu\nu}(t + t_0, \mathbf{x}) = b^2(t_0)g_{\mu\nu}(t, \mathbf{x}). \tag{5.16}$$

This is satisfied if

$$h(t + t_0) = b^2(t_0)h(t) \quad \text{and} \quad g'_{\mu\nu}(t + t_0, \mathbf{x}) = g'_{\mu\nu}(t, \mathbf{x}). \tag{5.17}$$

We demand that these relations hold for all t in order to preserve temporal equivalence. This holds if (not yet sure about "and only if")[1]

$$h(t) = e^{\frac{2t}{T}}, \quad g'_{\mu\nu}(t, \mathbf{x}) = g'_{\mu\nu}(\mathbf{x}), \quad b(t_0) = e^{\frac{t_0}{T}}. \tag{5.18}$$

We therefore have

$$ds^2 = e^{\frac{2t}{T}}(dt^2 - dx^2 - dy^2 - dz^2) = e^{\frac{2t}{T}}(dt^2 - dr^2 - r^2 d\Omega^2). \tag{5.19}$$

It is interesting to note the similarity of this metric with the FRW standard model de Sitter metric:

$$ds^2 = dt^2 - e^{\frac{2t}{T}}(dr^2 + r^2 d\Omega^2). \tag{5.20}$$

It appears that in hind sight one could have arrived at Eqn. (5.19) by simply noting first of all that up until the matter break away from the Hubble flow, the universe on the whole was evolving

[1] If the metric coefficients are constant in space we get spatial invariance in addition to temporal invariance. If the metric coefficients however were not spatial constants there would be gravitational gradients. Given that we see structure formation, in reality this nontrivial scenario must be the case. However given that it took a significant chunk of the age of the universe for matter to break away from the Hubble flow, such position dependence would be very mild. Given the zeroth order nature of out attempt to understand creation time, we will simply assume the metric coefficients are spatially constant.

according to the de Sitter FRW metric, and then extending the expansion of space to include also the expansion of time. From this point of view, one has set the lapse function equal to one (a gauge choice) in the usual standard model approach whereas here, one invokes the gauge choice of setting the lapse function equal to $e^{\frac{2t}{T}}$. From this point of view we have two gauge choices for expanding de Sitter spacetime. Hence the physics should still be the same. So we have not done anything theoretically violent here and so what we have done amounts simply to a change in how the theory is written down which has the advantage for our purposes of it becoming clearer how to calculate creation times. In other words, how to calculate ages based on the original non-expanded operational light clock. This convenient total contact with the standard model means that we do not need to worry about whether the theory agrees with experiment because it is in fact the result of a great many experiments and is the consensus view already!

From Eqn. (5.19) we can see that if we set $dr = d\Omega = 0$, $ds = cd\tau$ and $c = 1$ we have

$$\tau = \int_{-t_f}^{-t_0} e^{\frac{t}{T}} dt. \tag{5.21}$$

We would like to set a convention where we measure time from the beginning going forward so if we set $t \to -t$ and $dt \to -dt$ we get

$$\tau = \int_{t_0}^{t_f} e^{\frac{-t}{T}} dt. \tag{5.22}$$

Now we have to be careful because although there are symbols saying "t" which obviously are related to some sort of clock time, we need to make sure we have the correct t that we can identify with the original unexpanded creation clock. Firstly, we note that t is a comoving time, independent of expansion. The measured proper time of an observe that is going along with the expansion is related to the proper time parameter τ. However, there is a different non-expanding comoving time parameter that can be set up with each spatial hypersurface Σ_t. So we need to correctly set the scale between τ and the right-hand side in order to be able to identify our comoving time with the very first unexpanded comoving clock time. The other thing we should do is set the units of the original expanded clock. We do the latter by selecting

$$t = t'T\lambda, \quad dt = \lambda T dt', \quad \lambda \equiv \frac{\ln 2}{t_{1/2}} \tag{5.23}$$

to get (changing notation $t' \to t$)

$$\tau = \int_{t_0}^{t_f} \lambda T e^{-\lambda t} dt \tag{5.24}$$

where $t_{1/2}$ is one creation day and $\ln 2$ is the half life which is also present in RC circuits, LC circuits, radioactive decay and so on. To select the scale we need a current universal cosmic energy scale whereby we also know the corresponding initial scale on the first creation time spatial hypersurface. The obvious data to use is the CMB temperature T_{CMB} as the current universal energy scale and the proton-antiproton threshold pair production energy $T_{p\bar{p}}$ which gives the initial condition which makes operational light-clocks physically possible. Thus making the replacement (or more precisely, making the rescaling) $\lambda T \to \frac{T_{p\bar{p}}}{T_{CMB}}$ gives

$$\tau = \frac{T_{p\bar{p}}}{T_{CMB}} \int_{t_0}^{t_f} e^{-\lambda t} dt \tag{5.25}$$

which is precisely the equation previously written down by Schroeder where we now have the co-moving unexpanded time t correctly measured in unexpanded creation time days and the parameter τ correctly identifiable with our current time expanded proper time clocks.

We finish off this section with some theological remarks and other remarks.

Suppose that once creation was completed, God stopped expanding the heavens, which seems to be what is most naturally implied by the information in the Bible that is together with the expansion statements. Post-expansion, we would have $\Delta\tau = \Delta t$. We would have a totally smooth theologically problem-free transition from the sixth day of creation week to the seventh day. If the universe is still expanding, we would have to be more careful in our transition deductions from the sixth day of creation week onwards. Since creation week ended about 6000 years ago by our current clocks, and since redshift is observed at distance scales starting from beyond our local group which is about 10 million light years across, presumably it would take another few million years to notice whether or not the galaxy has stopped expanding.

As pointed out with the fractal bubble universe (i.e. our actual timescape cosmology universe composed of voids and walls which is void dominated by volume), which is the real universe that we actually live in, the average FRW type clock times represent average locations in the universe, which, in a void dominated universe such as our own, are located within the voids. This means that our earth clocks are running a bit slower with respect to that averaged clock, and hence the age of the universe in terms of our proper time should be more than 13.8 billion years. In terms of our calculation above, this would simply amount to the exponential function being replaced with a higher-order correction function such that when one integrates from 0 to 5.5, or from 0 to 6, one gets a local proper time age more along the lines of 15 billion years of age. This obviously in no way de-legitimizes or causes a problem for the paradigm presented in this chapter.

From the perspective of our current clocks, which scale like the creation clock, if we were somehow able to view creation week in such a way as to keep our own sense of time, we would have seen the Universe evolving and exciting creation events happening at super high speed.

To summarize in its simplest terms what we have done in deriving Schroeder's equation, it boils down to having taken the standard model of cosmology, chosen a different lapse function gauge choice, and carefully scaled the time parameters so as to give the correct interpretation in terms of us indeed having the starting unexpanded creation clock time t and us also having indeed the current earth-based proper time τ. It is a straight calculation which has no Biblical input and stands on its own independently of the claims of the Bible. Hence the results constitute strong evidence of the correctness of the Biblical claims that God created the heavens and the earth. Hence the heavens, and the laws of gravity applied to the stretching of the heavens in particular, declare the glory of God, just as the Bible indicates.

We finish off this section by commenting that in principle the calculation of the creation clock age of the universe could be improved by taking into account the fractal bubble age of the universe. However Schroeder's calculation is still a very good one because he effectively already has a correction built in to his calculation because of the apparent expansion acceleration of the universe. One could correct directly, as Schroeder has done, or else one could try to break the history of the universe up into an isotropic phase and a fractal bubble phase. The problem with doing this however is that it might be difficult to have an accurate CMB temperature for around 10 billion years ago and so any results would be more sketchy. This leaves us with Schroeder's equation as the best one currently known for converting the expansion proper time age of the universe into the creation comoving time for the age of the universe. Given that in the fractal bubble universe the lapse function differs between the interiors of walls and the centers of large voids by at most around 38%, and given that the galaxies only started breaking away from the Hubble flow about 10 billion years ago relative to expanding proper time, the true comoving creation time age of the universe would only be a very small correction. Certainly the correction would be under 10%, meaning that the actual time would have to be inside the range $5.5 < t_{cr} < 6$ days.

5.3 What the bible says about the universe and creation

In this study we examine the creation account which is mostly found in the first couple of chapters of Genesis. It is important that we let the Bible interpret itself in everything including the nature of time. Hence we need to approach the Bible like we know nothing and see exactly what the Bible says or else we will most likely not notice what the Bible says and hence not understand it.

The first thing that should be said is that when it comes to the Bible, the primary intended audience God had in mind was the people who live in the last days leading up to Jesus second coming. To see this point, consider 1 Corinthians 10:11, 2 Timothy 3:16, Daniel 12:4:

1 Corinthians 10:11. Now all these things (the context is a list of events and claims recorded in the Old Testament) **happened unto them for ensamples: and they are written for our admonition, upon whom the ends of the world are come.**

2 Timothy 3:16. All scripture is given by inspiration of God, and is profitable for doctrine, for reproof, for correction, for instruction in righteousness: Daniel 12:4. But thou, O Daniel, shut up the words, and seal the book, even to the time of the end: many shall run to and fro, and knowledge shall be increased.

We see that the Bible is good for doctrine. Doctrine means one's set of beliefs. The Bible is also good for correction of errors, both doctrinal and moral. What God did concerning creation is a doctrinal matter so the Bible is good for understanding creation. Moreover it is evident that God had the last days generation in mind when guiding the history of man because all of His major collective dealings with man have been recorded mainly for the last generation. God's audience definitely includes, in a primary sense, the last generation. Moreover we see that all Scripture is given by inspiration of God. In other words the Bible is His written communication to us. People do not speak to people without some intent. God spoke to us, primarily the last generation. It makes no sense for God to speak to the intent that we can never understand anything He says so what He says must in principle be understandable, and especially understandable to those living in the last days. This is confirmed in Daniel 12:4 where Gabriel says that knowledge shall be increased. The most direct context comes from the prior verse which speaks of they that be wise as turning many to righteousness. The wise must evidently therefore be disseminating knowledge of God's Word in general; not just the Bible prophecies which do not directly tell people what to do regarding how to be righteous or directly explain what righteousness looks like.

We now wish to further establish that God's primary intended audience regarding Him being the Creator, is the last generation. All of these things we are looking at are giving important clues which collectively will tell us how to understand Genesis chapter 1. To this end, consider Romans chapter 1:16-32:

Romans 1:16. For I am not ashamed of the gospel of Christ: for it is the power of God unto salvation to every one that believeth; to the Jew first, and also to the Greek.

Romans 1:17. For therein is the righteousness of God revealed from faith to faith: as it is written, The just shall live by faith.

Romans 1:18. For the wrath of God is revealed from heaven against all ungodliness and unrighteousness of men, who hold the truth in unrighteousness;

In another study we learned from the Bible the timing of God's wrath being revealed. God's wrath will be revealed starting with the first trumpet, but finishing in the full wrath of God unmixed with mercy in the seven last plagues immediately prior to Jesus arrival to take His saints to Heaven to judge the dead (and Satan and his angels) for 1000 years, after which they return to the earth and the wicked are resurrected to receive the executive judgment. The point here is that the primary audience of Romans chapter 1 aside from the direct people the letter was written to in Paul's day, is actually the last generation who are present on the earth when Jesus is going to come (after having completed His anti-typical work as our great High Priest in the Heavenly Sanctuary, the great original Sanctuary of which the Israelite Earthly Sanctuary was only a foreshadowing copy.).

Romans 1:19. Because that which may be known of God is manifest in them; for God hath shewed it unto them.

Romans 1:20. For the invisible things of him from the creation of the world are clearly seen, being understood by the things that are made, even his eternal power and Godhead; so that they are without excuse:

That which may be known of God is manifest to them. Who is them? It is the people living in the world in the last days. The scientifically literate generation. God has shown the truthfulness of His works in His creation. Romans 1:20 says that the final generation would clearly understand these works in nature concerning the creation of the world (which we have also just seen above is intimately tied in with the expansion of the heavens). In other words, God has designed that when Jesus comes, He will come to a world that full well knows that the Universe has been expanded and full well knows that the earth came into existence in the context of an expanding universe. God fully intends for the last generation to have a much greater knowledge of science than at any previous time in history. People then, with a much greater scientifically based knowledge through which to understand both the Bible and the book of nature, will be totally without excuse when Jesus comes. When scientific knowledge is brought to a sense of fullness, God is vindicated. God can only be denied on grounds of partial knowledge's mixed with errors in science, held up by the proud as being the high standard of truth. But God says to push on to a more full and complete knowledge, and such honest full and complete knowledge will vindicate the truths of His Word. This is important because if people do not see God's Word as Truth they will not then follow the counsel contained therein which would save them from eternal death and fit them for eternal life and to become acquainted on eternal friendly terms with their Creator.

Romans 1:21. Because that, when they knew God, they glorified him not as God,

neither were thankful; but became vain in their imaginations, and their foolish heart was darkened.

Romans 1:22. Professing themselves to be wise, they became fools,

The immediate context regarding professing to be wise is verse 20 concerning understanding God's creation. In other words Romans 1:22 is talking primarily about last day people proudly presenting themselves as having superior scientific knowledge and arrogantly flaunting their incomplete and in places perverted understanding of science in such a way as to try and justify the rejection of God. Psalms 14:1 says

Psalms 14:1. The fool hath said in his heart, There is no God. They are corrupt, they have done abominable works, there is none that doeth good.

According to Romans chapter 1, the last generation will contain many scientifically semi-literate atheists who have said in their heart (in other words they actually believe their professed atheism to be correct) there is no God and publicly based their claims on their partial understanding of science which they present as overwhelmingly complete for the purposes of denying God. Their self-destructive cause is aided by the fact that on the whole the Bible believing Christians are also at most scientifically semi-literate.

Romans 1:23. And changed the glory of the uncorruptible God into an image made like to corruptible man, and to birds, and fourfooted beasts, and creeping things.

In other words the atheists replaced the creation account with a story that puts corrupt men (i.e. hominids) and animals in the place of God as being instrumental in the arrival of man. In other words a story of macro evolution is prophesied to be taught in the last days while the judgment is going on in Heaven.

Romans 1:24. Wherefore God also gave them up to uncleanness through the lusts of their own hearts, to dishonour their own bodies between themselves:

Romans 1:25. Who changed the truth of God into a lie, and worshipped and served the creature more than the Creator, who is blessed for ever. Amen.

The context of this is obviously surrounding the issue of whether God exists and whether God is the Creator. Hence this is a prophesy that last day morally corrupt people would take the very scientific evidences which vindicate God, and pervert those evidences to support an alternative and Godless creation narrative. Hence replace the truth with a lie. We see here a prophesy that the creation account would be replaced with an evolution account which elevates animals when it comes to the origins of man in place of recognizing God in the creation of man.

Romans 1:26. For this cause God gave them up unto vile affections: for even their women did change the natural use into that which is against nature:

Romans 1:27. And likewise also the men, leaving the natural use of the woman,

burned in their lust one toward another; men with men working that which is unseemly, and receiving in themselves that recompence of their error which was meet.

Romans 1:28. And even as they did not like to retain God in their knowledge, God gave them over to a reprobate mind, to do those things which are not convenient;

Romans 1:29. Being filled with all unrighteousness, fornication, wickedness, covetousness, maliciousness; full of envy, murder, debate, deceit, malignity; whisperers,

Romans 1:30. Backbiters, haters of God, despiteful, proud, boasters, inventors of evil things, disobedient to parents,

Romans 1:31. Without understanding, covenantbreakers, without natural affection, implacable, unmerciful:

Romans 1:32. Who knowing the judgment of God, that they which commit such things are worthy of death, not only do the same, but have pleasure in them that do them.

We see here that the atheist movement would evolve into an LGBT supporting movement. This movement is also prophesied to go under a name other than "vile." The identity of these vile ones is spelt out in the end time prophecy in Isaiah chapter 32. Consider Isaiah 32:4-5:

Isaiah 32:4. The heart also of the rash shall understand knowledge, and the tongue of the stammerers shall be ready to speak plainly.

Isaiah 32:5. The vile person shall be no more called liberal, nor the churl said to be bountiful.

So we see that in the last days the vile ones will be known in the world as the liberal ones. When the judgments of God are poured out on the vile, the tongues of God's people will speak plainly and refer to the vile as vile, rather than referring to the vile by the last generation euphemism of "liberal." It is interesting to note that according to Romans 1:31 these people are without understanding.

To finish setting the stage to properly understand the creation account in Genesis, we will first look at a critically important chapter in Psalms followed by a look at what God says together with the repeated claim in the Old Testament that God stretched out the heavens. So firstly, consider Psalms chapter 19:

Psalms 19:1. (To the chief Musician, A Psalm of David.) The heavens declare the glory of God; and the firmament sheweth his handywork.

We see that the universe is instrumental in speaking to us of God's works. Thus God has not left us solely with the Bible in order to see God's creative truth. God's creative works are being proclaimed from the heavens. The dim in understanding (remember that we have just seen that the ones who have no understanding are the very ones who profess to be full of scientific understanding and who have dismissed God, replaced the truth with evolution, and have gone on to support an LGBT agenda, becoming vile under the guise of being "liberal.") are not able to understand what they see. To see who are the ones who are capable of correctly recognizing the declarations of God's glory emanating from the heavens, we need to keep reading the chapter.

Psalms 19:2. Day unto day uttereth speech, and night unto night sheweth knowledge.

God lets us know that the heavens reveal scientific knowledge. In context, this scientific knowledge must be directly relevant to the declarations of God's glory. We therefore should seek to understand scientific knowledge of the heavens.

Psalms 19:3. There is no speech nor language, where their voice is not heard.

Science transcends language and national boundaries.

Psalms 19:4. Their line is gone out through all the earth, and their words to the end of the world. In them hath he set a tabernacle for the sun,

Psalms 19:5. Which is as a bridegroom coming out of his chamber, and rejoiceth as a strong man to run a race.

Psalms 19:6. His going forth is from the end of the heaven, and his circuit unto the ends of it: and there is nothing hid from the heat thereof.

Obviously these two verses are given from the perspective of the earth based observers.

Psalms 19:7. The law of the LORD is perfect, converting the soul: the testimony of the LORD is sure, making wise the simple.

In context, we see that the Law of The Lord has something to do with being able to understand the declarations of the heavens.

Psalms 19:8. The statutes of the LORD are right, rejoicing the heart: the commandment of the LORD is pure, enlightening the eyes.

To receive the declarations from the heavens we use our eyes (whether human eyes or telescopes). The eyes are only intelligently digesting what they are seeing if they have been enlightened by God's commandments. Unenlightened eyes are left to grope in darkness regarding what they are seeing in the heavens. Eyes have they, but they see not. Given our physics observations in this document the truth of all this should now be more obvious.

Psalms 19:9. The fear of the LORD is clean, enduring for ever: the judgments of the LORD are true and righteous altogether.

Psalms 19:10. More to be desired are they than gold, yea, than much fine gold:

sweeter also than honey and the honeycomb.

Psalms 19:11. Moreover by them is thy servant warned: and in keeping of them there is great reward.

Psalms 19:12. Who can understand his errors? cleanse thou me from secret faults.

Psalms 19:13. Keep back thy servant also from presumptuous sins; let them not have dominion over me: then shall I be upright, and I shall be innocent from the great transgression.

Psalms 19:14. Let the words of my mouth, and the meditation of my heart, be acceptable in thy sight, O LORD, my strength, and my redeemer.

We have learnt from Psalms chapter 19 that scientific knowledge is available every day and night regarding the heavens, from the heavens, and that this knowledge can be correctly understood by eyes that have been enlightened by God's commandments. Moreover it has been claimed that the information available for our learning from the heavens is declaring God's glory.

It makes sense therefore, to examine other parts of the Bible and see if there are any particular aspects of the heavens that we should pay particular attention to when contemplating the divine message contained therein for us.

Hence, before we look at Genesis, we first take note that the Bible says God stretched out the heavens and we note what else is said in conjunction with this. Consider Job 9:7-10, Psalms 104:1-2, Isaiah 40:22, Isaiah 42:5, Isaiah 44:24, Isaiah 45:12, Isaiah 48:13, Isaiah 51:13, Jeremiah 10:12, Jeremiah 51:15 and Zechariah 12:1:

Job 9:7. Which commandeth the sun, and it riseth not; and sealeth up the stars.

This is a very interesting verse. We will defer further comment on it until we look at the creation of the stars in Genesis chapter 1.

Job 9:8. Which alone spreadeth out the heavens, and treadeth upon the waves of the sea.

Here we see that the heavens got spread out. The implication is that they were once not spread out. Notice also that Job is apparently the oldest book in the Bible and it speaks of the expansion of the universe. Modern scientists have only come into belief in this Biblical doctrine since some time during the 20th century.

Job 9:9. Which maketh Arcturus, Orion, and Pleiades, and the chambers of the south.

Job 9:10. Which doeth great things past finding out; yea, and wonders without number.

Psalms 104:1.Bless the LORD, O my soul. O LORD my God, thou art very great; thou art clothed with honour and majesty.

Psalms 104:2. Who coverest thyself with light as with a garment: who stretchest out the heavens like a curtain:

This is a revealing verse. Firstly, we note that "thyself" is a supplied word in the King James English translation. Hence this word is not actually in the verse. So God covered the universe with light, and this was directly connected with stretching out the heavens. The Bible thus speaks of the expansion of the Universe, and this is done in conjunction with the covering of the heavens with light. This is a very important point that should never be forgotten.[2]

Isaiah 40:22. It is he that sitteth upon the circle of the earth, and the inhabitants thereof are as grasshoppers; that stretcheth out the heavens as a curtain, and spreadeth them out as a tent to dwell in:

The expansion of the universe is mentioned here again. This time it is mentioned in conjunction with the heavens being spread out, and to emphasize the implication about the universe being very big as a result of the expansion, God likens the inhabitants of the earth to grasshopers. The notion of the tent would seem to suggest, although not conclusively so, that the expansion of the universe stopped with the completion of creation. One has to be careful on this point. The scale of observed redshift is beyond 10 million light years so if the universe has stopped expanding 6000 years ago, we would not be able to tell.

Isaiah 42:5. Thus saith God the LORD, he that created the heavens, and stretched them out; he that spread forth the earth, and that which cometh out of it; he that giveth breath unto the people upon it, and spirit to them that walk therein:

Here we see that the expansion of the universe is also directly connected with the spreading forth of the earth. The obvious implication here is that before the expansion the earth had not been spread forth. We therefore see that the earth was not in any kind of formed state in the beginning. The earth came into being during the expansion of the universe. Moreover, if a material gets spread, it follows that the material exists at least by the time that the spreading is to commence. Hence the materials which composed the earth predate the earth according to the Bible, and only came to define what we recognize to be the earth during the expansion of the universe.

Isaiah 44:24. Thus saith the LORD, thy redeemer, and he that formed thee from the womb, I am the LORD that maketh all things; that stretcheth forth the heavens alone; that spreadeth abroad the earth by myself (past tense);

Again we see that God expanded the universe and formed the earth in connection with the expansion of the universe and not before. Also, the way this comes across in English gives the impression of present continual tense as if the universe is still expanding. However the same applies here to the earth being spread forth, but since we know that to be past tense, one again wonders whether the implication here is that the universe has ceased to be expanding.

[2] Jumping prematurely ahead, the reason this should not be forgotten is that the stretching of the heavens also stretched the clock times out, thus giving the light all the time needed to traverse the vast distances in space.

Isaiah 45:12. I have made the earth (past tense), and created man upon it (past tense): I, even my hands, have stretched out the heavens, and all their host have I commanded (past tense).

This verse contains more important information. We see here that in conjunction with the expansion of the universe, the matter structures are conforming to God's Law. The specific law that guides the host in the heavens is gravity. So we have an important clue here in that this verse has directly focused attention on the expansion of the universe, gravity and God's Law. Recall from earlier that it is the Law of God that enlightens the eyes regarding the heavens declaration of God's glory. We are thus directed by God to consider gravity very carefully in conjunction with the expansion of the universe. It will turn out that doing precisely what the Bible says to do proves to be the key which vindicates the creation account in the book of Genesis, and hence provides the discernment concerning the heavens declaring God's glory. Curiously, the making of the earth is past tense, and the applying of the law of gravity is past tense, and sandwiched here in the middle is the expansion of the universe, which from a normal reading comprehension point of view would seem as though it should be past tense also.

Isaiah 48:13. Mine hand also hath laid the foundation of the earth (past tense), and my right hand hath spanned the heavens: when I call unto them, they stand up together.

Once again we see a past tense action (the laying of the foundation of the earth) and another past tense action (the heavens standing up together) with the expansion of the universe being the inbetween thing on the list. Again, the usual way of reading in English would give the impression that the expansion is also past tense.

Isaiah 51:13. And forgettest the LORD thy maker, that hath stretched forth the heavens, and laid the foundations of the earth (past tense); and hast feared continually every day because of the fury of the oppressor, as if he were ready to destroy? and where is the fury of the oppressor?

The making of man is past tense, as is the laying of the foundations of the earth so once again the most natural and obvious thing to conclude as an exercise in reading comprehension is that the expansion of the universe is also past tense.

Jeremiah 10:12. He hath made the earth by his power (past tense), he hath established the world by his wisdom (past tense), and hath stretched out the heavens by his discretion.

Once again the things on the list that includes the expansion of the universe is past tense so one would again get the impression that the expansion is past tense.

Jeremiah 51:15. He hath made the earth by his power (past tense), he hath established the world by his wisdom (past tense), and hath stretched out the heaven by his understanding.

Evidently it took understanding to expand the universe. The Bible thus makes plain that God is the ultimate Physicist. He understands the intricacies of physics and cosmology in any amount of necessary detail to do what He has done. And once again God mentions the expansion of the universe in conjunction with other things that are definitely past tense, indicating that the first thing we should consider is that the expansion of the universe has ceased.

Zechariah 12:1. The burden of the word of the LORD for Israel, saith the LORD, which stretcheth forth the heavens, and layeth the foundation of the earth (past tense), and formeth the spirit of man within him (past tense).

We now summarize some key points before looking at Genesis:

1. At least eleven times the Bible emphasizes the expansion of the universe.

2. The Bible explains that the heavens night after night declare God's glory.

3. The Bible explains that in order to see this, people have to have their eyes enlightened by God's Law.

4. To those whose eyes have been enlightened, they are to direct their attention to gravity in connection with the expansion of the Universe and focus on this in particular, in order to be able to discern the declaration of God's glory by the heavens.[3]

5. The ones who have rejected God have been left in darkness. The last generation before God's judgments fall are prophesied to be atheist, macro evolution believers, liberal, and LGBT supportive. The development and maturing of the LGBT supportive aspect is the last prophesied thing in Romans chapter 1 before the judgments of God are seen falling in the earth. Since all has been fulfilling exactly in accordance with Bible prophecy and since we see the very last stages of the development of Biblically defined evil manifesting in the world, it stands to reason that the next big development will be the falling of God's judgments in the world.

6. The expansion of the universe is usually included in a list of God's works in which the elements of the list are past tense, which gives the natural impression that God our Creator is claiming that the expansion of the universe is also past tense.

We now turn to the creation account in the book of Genesis.

Genesis 1:1. In the beginning God created the heaven and the earth.

Notice that there is some physical state called "the beginning" which is associated with the materials that define the heaven and the earth. Since we have already learnt that the earth doesn't get discernibly formed until during the expansion of the universe, it follows that in the beginning all of the materials which compose the heavens and the earth were created.

[3]Getting ahead of ourselves, it turns out that when one does this one sees that the standard cosmological age of the universe, given in terms of stretching time, is the same as 6 days according to non-stretching time. The calculation is totally independent of anything from the Bible so this is indeed a strong vindication of the Biblical account being true, just as written.

Genesis 1:2. And the earth was without form, and void; and darkness was upon the face of the deep. And the Spirit of God moved upon the face of the waters.

The earth was without form and void. If the earth is without form then by modern definition it does not exist. If the earth had say a spherical form then it would have form and may or may not be void. But that's not what the Bible says. The Bible says that the earth itself was without form. In other words the materials that would be used to assemble the earth was not yet put together in a coherent way. If the earth is without form then the materials are evidently not descernable from the non-earth materials that define the heavens. Moreover, there is darkness. The heaven therefore must be in a state whereby there is no light moving around. We have learnt that this must be referring to the state of the universe before there was an expansion describable in terms of light-based clocks. Remember that one of the things above that is connected with the expansion of the universe is that the heavens are clothed with light. Hence if the universe is dark, it must be pre-light-clock expansion. From a physics point of view this must imply that there exist no operational light clocks which is only the case if the temperature is too hot for stable proton-anti-proton pair production. The threshold temperature for this is 10.9 trillion degrees kelvin. It might initially be considered a speculation that the temperature was very high when the heaven and earth materials was in this dark formless void state, but at this point we note from other parts of the Bible that God stretched out the heavens. Physically, expansion cools the universe. So when the heavens were not sufficiently expanded, the temperature was too high for light-clocks to exist and hence the universe was dark. Moreover, since light-clocks are what operationally define our concepts of time, it follows that in this dark initial state there was no conventional operational notion of time. Hence the universe, once created, was in a timeless formless dark and very hot state outside the domain of validity of models extrapolating FRW based cosmologies back in time to see how the universe evolved. Notice also that in the Hebrew language the only word available to describe any kind of liquid type behavior of any physical/chemical composition was "water." So here liquid water cannot necessarily be assumed.

Genesis 1:3. And God said, Let there be light: and there was light.

In order for light to propagate around, the universe must have cooled to a temperature less than or equal to 10.9 trillion degrees kelvin. This is accomplished by stretching the heavens. So God started stretching the heavens so at this point in Genesis we are now looking at an expanding universe.

Genesis 1:4. And God saw the light, that it was good: and God divided the light from the darkness.

Genesis 1:5. And God called the light Day, and the darkness he called Night. And the evening and the morning were the first day.

Light started being able to propagate out through an increasingly large expanse of space.

Genesis 1:6. And God said, Let there be a firmament in the midst of the waters, and let it divide the waters from the waters.

Genesis 1:7. And God made the firmament, and divided the waters which were under the firmament from the waters which were above the firmament: and it was so.

Genesis 1:8. And God called the firmament Heaven. And the evening and the morning were the second day.

Structure formation Stars and galaxies created. The earth took some kind of form. Notice also that in the Hebrew language the only word available to describe any kind of liquid type behavior of any physical/chemical composition was "water." So here liquid water cannot necessarily be assumed. What we see here in day two are the giant gas clouds coming together getting ready for God's use to form structure.

Genesis 1:9. And God said, Let the waters under the heaven be gathered together unto one place, and let the dry land appear: and it was so.

With the third day, we have another mode of darkness (which when coupled to another "light" forms another, i.e. a third, creation day), this time from the point of view of the earth that is being formed within the gas cloud.

Genesis 1:10. And God called the dry land Earth; and the gathering together of the waters called he Seas: and God saw that it was good.

Here we see the spreading forth of the earth. In other words here we see the creation of the earth.

Genesis 1:11. And God said, Let the earth bring forth grass, the herb yielding seed, and the fruit tree yielding fruit after his kind, whose seed is in itself, upon the earth: and it was so.

Genesis 1:12. And the earth brought forth grass, and herb yielding seed after his kind, and the tree yielding fruit, whose seed was in itself, after his kind: and God saw that it was good.

Genesis 1:13. And the evening and the morning were the third day.

The earth was mature. Grass and trees were created.

Genesis 1:14. And God said, Let there be lights in the firmament of the heaven to divide the day from the night; and let them be for signs, and for seasons, and for days, and years:

The morning part of the third day evidently arrived when the gas cloud cleared and the light from the stars could penetrate and be seen from the perspective of the earth.

Genesis 1:15. And let them be for lights in the firmament of the heaven to give light upon the earth: and it was so.

Genesis 1:16. And God made two great lights; the greater light to rule the day, and the lesser light to rule the night: he made the stars also.

Genesis 1:17. And God set them in the firmament of the heaven to give light upon the earth,

Genesis 1:18. And to rule over the day and over the night, and to divide the light from the darkness: and God saw that it was good.

Genesis 1:19. And the evening and the morning were the fourth day.

At this point we recall Job 9:7 about the stars being sealed up. Taking this into account, one might expect that what is meant in Genesis regarding stars is that they were created earlier but they were not discernible until the gas cloud cleared in day 4.

Genesis 1:20. And God said, Let the waters bring forth abundantly the moving creature that hath life, and fowl that may fly above the earth in the open firmament of heaven.

Genesis 1:21. And God created great whales, and every living creature that moveth, which the waters brought forth abundantly, after their kind, and every winged fowl after his kind: and God saw that it was good.

Genesis 1:22. And God blessed them, saying, Be fruitful, and multiply, and fill the waters in the seas, and let fowl multiply in the earth.

Genesis 1:23. And the evening and the morning were the fifth day.

Evidently God then created the sea life etc. As to the question of what exactly did it look like to see the sea life "brought forth" from the waters, the Bible does not appear to say. If the Bible does not comment on something it would be silly to make a controversy over it. If God hasn't commented on something then there is no concrete claim to argue over. Evidently the important thing here is that we now see the sea life and the birds etc. Apparently the Hebrew word that got translated by the transators into "great whales" actually means "great reptiles" which presumably could predominantly be referring to dinosaurs. Moreover, apparently the winged fowl is actually referring to the flying insects.

Genesis 1:24. And God said, Let the earth bring forth the living creature after his kind, cattle, and creeping thing, and beast of the earth after his kind: and it was so.

Genesis 1:25. And God made the beast of the earth after his kind, and cattle after their kind, and every thing that creepeth upon the earth after his kind: and God saw that it was good.

The creeping thing presumably would be insects. A diversity of creatures were brought forth from the earth, each creature after his kind. There seems to be a clue here that "brought forth" means to appear in the sense of reproducing to populate, each after its kind. How the initial creature of each kind appeared is not explicitly spelt out. Evidently God simply expects us to believe that He made it so somehow.

Genesis 1:26. And God said, Let us make man in our image, after our likeness: and let them have dominion over the fish of the sea, and over the fowl of the air, and over the cattle, and over all the earth, and over every creeping thing that creepeth upon the earth.

Genesis 1:27. So God created man in his own image, in the image of God created he him; male and female created he them.

Genesis 1:28. And God blessed them, and God said unto them, Be fruitful, and multiply, and replenish the earth, and subdue it: and have dominion over the fish of the sea, and over the fowl of the air, and over every living thing that moveth upon the earth.

Genesis 1:29. And God said, Behold, I have given you every herb bearing seed, which is upon the face of all the earth, and every tree, in the which is the fruit of a tree yielding seed; to you it shall be for meat.

Genesis 1:30. And to every beast of the earth, and to every fowl of the air, and to every thing that creepeth upon the earth, wherein there is life, I have given every green herb for meat: and it was so.

Genesis 1:31. And God saw every thing that he had made, and, behold, it was very good. And the evening and the morning were the sixth day.

Genesis 2:1. Thus the heavens and the earth were finished, and all the host of them.

We thus come to the end of the creating of creation. Upon completing creation God apparently stopped stretching out the heavens. So at the end of day 6, time stopped getting stretched out so day 7 was the first day that felt like a 24 hour day to an internal matter-based observer in the universe.

Genesis 2:2. And on the seventh day God ended his work which he had made; and he rested on the seventh day from all his work which he had made.

Genesis 2:3. And God blessed the seventh day, and sanctified it: because that in it he had rested from all his work which God created and made.

Genesis 2:4. These are the generations of the heavens and of the earth when they were created, in the day that the Lord God made the earth and the heavens,

Notice here that a plurality of generations were brought forth over two days of creation time.

This again signals some kind of difference between creation time and current earth rotation time which is the same as what one might call zero-expansion-rate time. The reader is reminded that earlier in this study we saw from the Bible that this particular topic of analysis is the Bible ordained topic which should bring resolution to the question of how the heavens declare the glory of God. Interestingly, the great connecting time between creation and the generations of man is the Sabbath day. The Sabbath is very naturally the visible sign distinguishing between those who worship and obey The Creator and those who do not.

Genesis 2:5. And every plant of the field before it was in the earth, and every herb of the field before it grew: for the Lord God had not caused it to rain upon the earth, and there was not a man to till the ground.

Genesis 2:6. But there went up a mist from the earth, and watered the whole face of the ground.

Genesis 2:7. And the Lord God formed man of the dust of the ground, and breathed into his nostrils the breath of life; and man became a living soul.

Notice here that God does not say that Adam was brought forth from the beasts of the earth, but in contrast, was formed from the dust in the ground. We see here that we distinctly have creation, and not gradual trans-species evolution. Macro evolution is thus inherently unbiblical and can therefore be clearly contested as being what it is, false. Moreover the reader is referred back to the prophecy in Romans chapter 1 concerning the last days, and how the theory of evolution would be developed and marketed in defiance of God.

Genesis 2:8. And the Lord God planted a garden eastward in Eden; and there he put the man whom he had formed.

Genesis 2:9. And out of the ground made the Lord God to grow every tree that is pleasant to the sight, and good for food; the tree of life also in the midst of the garden, and the tree of knowledge of good and evil.

Genesis 2:10. And a river went out of Eden to water the garden; and from thence it was parted, and became into four heads.

Genesis 2:11. The name of the first is Pison: that is it which compasseth the whole land of Havilah, where there is gold;

Genesis 2:12. And the gold of that land is good: there is bdellium and the onyx stone.

Genesis 2:13. And the name of the second river is Gihon: the same is it that compasseth the whole land of Ethiopia.

Genesis 2:14. And the name of the third river is Hiddekel: that is it which goeth toward the east of Assyria. And the fourth river is Euphrates.

Genesis 2:15. And the Lord God took the man, and put him into the garden of Eden to dress it and to keep it.

Genesis 2:16. And the Lord God commanded the man, saying, Of every tree of the garden thou mayest freely eat:

Genesis 2:17. But of the tree of the knowledge of good and evil, thou shalt not eat of it: for in the day that thou eatest thereof thou shalt surely die.

Genesis 2:18. And the Lord God said, It is not good that the man should be alone; I will make him an help meet for him.

Genesis 2:19. And out of the ground the Lord God formed every beast of the field, and every fowl of the air; and brought them unto Adam to see what he would call them: and whatsoever Adam called every living creature, that was the name thereof.

Here we see the creation account of Eve. The creation of Eve clearly has nothing to do with evolution. Likewise, we see that every beast of the field was formed out of the ground also. This is in contrast to beasts of the field evolving from progressively more primitive life forms. Each individual species was created out of the ground. The Bible does not say that one life form was formed out of the ground and that all other species emanated from that life form that is now distinct from the ground. Again, macro evolution is contested (and moreover condemned in Bible prophecy speaking of its rise in the last days).

Genesis 2:20. And Adam gave names to all cattle, and to the fowl of the air, and to every beast of the field; but for Adam there was not found an help meet for him.

Genesis 2:21. And the Lord God caused a deep sleep to fall upon Adam, and he slept: and he took one of his ribs, and closed up the flesh instead thereof;

Genesis 2:22. And the rib, which the Lord God had taken from man, made he a woman, and brought her unto the man.

Genesis 2:23. And Adam said, This is now bone of my bones, and flesh of my flesh: she shall be called Woman, because she was taken out of Man.

Genesis 2:24. Therefore shall a man leave his father and his mother, and shall cleave unto his wife: and they shall be one flesh.

Genesis 2:25. And they were both naked, the man and his wife, and were not ashamed.

Notice here that the very institution of marriage is a rendering of the Biblical claim that Eve was created from Adam's body and hence is one flesh with her husband. It is therefore natural in the Romans chapter 1 prophesied rejection of God in the last days, that after the theory of evolution takes firm hold in people's minds, that movements would arise which would progressively attack the institution of Biblically defined marriage! Hence the natural outgrowth of the

LGBT movement from militant atheism. Also hence the reason why LGBT type defiance is the last general kind of defiance mentioned in Romans 1 to precede the falling of the judgments of God. We close the Bible study part of the study with an interesting subset of a last days prophesy (to see that Deuteronomy chapter 32 is a last days prophesy one need only read chapter 31 to get the context and lead in to chapter 32) in Deuteronomy chapter 32. Consider Deuteronomy 32:1-7:

Deuteronomy 32:1. Give ear, O ye heavens, and I will speak; and hear, O earth, the words of my mouth.

Deuteronomy 32:2. My doctrine shall drop as the rain, my speech shall distil as the dew, as the small rain upon the tender herb, and as the showers upon the grass:

Evidently last day related present truth is accounted as the result of the falling of the latter rain.

Deuteronomy 32:3. Because I will publish the name of the LORD: ascribe ye greatness unto our God.

Deuteronomy 32:4. He is the Rock, his work is perfect: for all his ways are judgment: a God of truth and without iniquity, just and right is he.

Deuteronomy 32:5. They have corrupted themselves, their spot is not the spot of his children: they are a perverse and crooked generation.

Deuteronomy 32:6. Do ye thus requite the LORD, O foolish (recall that a fool has said in his heart that there is no God, so the foolish, by Biblical reckoning is probably a reference to atheists, and more precisely, atheists who have opportunity to know better but have chosen to set themselves apart from God) people and unwise? is not he thy father that hath bought thee? hath he not made thee, and established thee?

Notice that the problem here is the same problem as prophesied in Romans chapter 1 about the rejection of God, the rise of atheism and the resulting floodgates of iniquity being opened upon the world.

Deuteronomy 32:7. Remember the days of old, consider the years of many generations: ask thy father, and he will shew thee; thy elders, and they will tell thee.

Notice that God is trying to call the atheist liberals attention to the days of old. Given the nature of the problem which provides the context to this verse, this must be the days of creation week. And also the years of many generations. This is the generations of man, that God has had a very active part in being involved with (for example consider Noah's flood, the tower of Babel, the fall of Egypt, the fall of Babylon, Media-Persia, Greece, Pagan Rome, Papal Rome, the dilution of Papal Rome by secularism and so on). Here we see a distinction between "creation clock time" and "generations of man earth rotation clock time." The Bible also defines a third clock, heaven time, which currently ticks such that one day according to heaven time is equivalent to 1000 years of earth rotation clock time. To see this, first consider Psalms 90:3-4:

Psalms 90:3. Thou turnest man to destruction; and sayest, Return, ye children of men.

Psalms 90:4. For a thousand years in thy sight are but as yesterday when it is past, and as a watch in the night.

This is a warning about the coming day of The Lord, which according to heaven time takes place in one day, which by earth rotation time is 1000 years. To see this more clearly, consider 2 Peter 3:3-10 and Revelation 20:1-5:

2 Peter 3:3. Knowing this first, that there shall come in the last days scoffers, walking after their own lusts,

2 Peter 3:4. And saying, Where is the promise of his coming? for since the fathers fell asleep, all things continue as they were from the beginning of the creation.

2 Peter 3:5. For this they willingly are ignorant of, that by the word of God the heavens were of old, and the earth standing out of the water and in the water:

2 Peter 3:6. Whereby the world that then was, being overflowed with water, perished:

2 Peter 3:7. But the heavens and the earth, which are now, by the same word are kept in store, reserved unto fire against the day of judgment and perdition of ungodly men.

2 Peter 3:8. But, beloved, be not ignorant of this one thing, that one day is with the Lord as a thousand years, and a thousand years as one day.

2 Peter 3:9. The Lord is not slack concerning his promise, as some men count slackness; but is longsuffering to us-ward, not willing that any should perish, but that all should come to repentance.

2 Peter 3:10. But the day of the Lord will come as a thief in the night; in the which the heavens shall pass away with a great noise, and the elements shall melt with fervent heat, the earth also and the works that are therein shall be burned up.

Revelation 20:1. And I saw an angel come down from heaven, having the key of the bottomless pit and a great chain in his hand.

Revelation 20:2. And he laid hold on the dragon, that old serpent, which is the Devil, and Satan, and bound him a thousand years,

Revelation 20:3. And cast him into the bottomless pit, and shut him up, and set a seal upon him, that he should deceive the nations no more, till the thousand years should be fulfilled: and after that he must be loosed a little season.

Revelation 20:4. And I saw thrones, and they sat upon them, and judgment was given unto them: and I saw the souls of them that were beheaded for the witness of Jesus, and for the word of God, and which had not worshipped the beast, neither his image, neither had received his mark upon their foreheads, or in their hands; and they lived and reigned with Christ a thousand years.

Revelation 20:5. But the rest of the dead lived not again until the thousand years were finished.

In summary, we note from the Bible that:

1. God expanded the universe (and according to a surface level reading of the English, has also apparently ceased expanding the universe since His completion of creation),
2. The Bible uses a creation clock,
3. The Bible uses an earth rotation based clock,
4. The Bible uses a heaven based clock.

The Bible thus teaches the existence of a relativity of time. The Bible declares the relationship between earth rotation time and heaven time, but the Bible does not explicitly say how creation time is related to earth rotation time and hence also heaven time. However the Bible does tell us that the heavens are speaking to us and that in order to discern the anti-atheistic glory of God, we are to focus particular attention on the law of gravity in connection with the expansion of the universe. The application of the laws of gravity to the expansion of the universe actually gives us the relationship between creation time and earth rotation time. The startling result is that secular cosmology has already shown Genesis chapter 1 to be literally true in terms of its time frame of events. Notice also that the Bible teaches clearly that there was no death before sin. Natural selection is thus automatically ruled out by the Bible.

Now we will put in the specific information from the Genesis account into the expansion clock timeline:

Day 6: From 6000 (unexpanded years) to 222.2 million years ago.
Genesis account: Humans, animals, cows, horses, general cattle beasts. Creeping things (varous creeping insects presumably).
Compare this with the standard evolutionary timeline: 200,000 years ago: Homo sapiens in Africa. 10 million years ago: Diversity of insects, first large horses. 15 million years ago: mastodons (bison, buffalo, antelopes, gazelles, sheep, goats, cows etc), bovids and kangaroos.

Day 5: From 222.2 million years ago to 666.5 million years ago.
Genesis account: Abundant water life. Flying insects. Dinosaurs/reptiles.
Compare this with the standard evolutionary timeline: 220 million years ago: First crocodilians and flies. 225 million years ago: First dinosaurs. Teleosti (96% of all extant species of fish. Ray-fined fishes). 245 million years ago: ichthyosaurs (large marine reptiles).

Day 4: From 666.5 million years ago to 1.555 billion years ago.

Genesis account: The stars and galaxies become visible.
According to evolutionary theory land plants are thought to have evolved from green algae around 850 million years ago and algae like plants around 1 billion years ago.

Day 3: From 1.555 billion years ago to 3.3318 billion years ago.
Genesis account: The Earth was mature. Grass and trees appeared.

Day 2: From 3.3318 billion years ago to 6.88497 billion years ago.
Genesis account: Structure formation. Stars and galaxies. The earth took some kind of form.

Day 1: From 6.99497 billion years ago to 13.99 billion years ago.
Genesis account: Large expanse was created. Light started to propagate.

We are not here endorsing of evolution. This is merely an observation that timelines are broadly in harmony between the evolutionary model and the Biblical creation account. The point of pointing this out is that even if one insists that macro evolution happened, given that the Bible is silent on precisely how God brought forth non-human species, one cannot appeal to ones science beliefs as any sort of reason for dismissing the Bible. Not only is there no reason to dismiss the Bible, there are an abundance of reasons to accept the Bible. Here we have covered the foundational physics and cosmology aspects of the evidences from the category of natural sciences. An abundance of other reasons exist, spanning the other categories of science as well as verifiable history recorded in the Bible, including history which contains supernatual events, and also verifiable fulfillment of detailed event and time specific Bible prophecy written up to the order of 2,500 years in advance. The next volume of Evidence for the Bible concentrates on Bible prophecy.

Appendix A

Sedenion Tables

Table A.1: Multiplication Table for $\mathbf{e}_i\mathbf{e}_j \in \mathbb{S}$

$\mathbf{e}_i\mathbf{e}_j$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$
$\mathbf{e}_8$	$\mathbf{e}_8$	$-\mathbf{e}_9$	$-\mathbf{e}_{10}$	$-\mathbf{e}_{11}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_{14}$	$-\mathbf{e}_{15}$
$\mathbf{e}_9$	$\mathbf{e}_9$	$\mathbf{e}_8$	$-\mathbf{e}_{11}$	$\mathbf{e}_{10}$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{14}$
$\mathbf{e}_{10}$	$\mathbf{e}_{10}$	$\mathbf{e}_{11}$	$\mathbf{e}_8$	$-\mathbf{e}_9$	$-\mathbf{e}_{14}$	$-\mathbf{e}_{15}$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$
$\mathbf{e}_{11}$	$\mathbf{e}_{11}$	$-\mathbf{e}_{10}$	$\mathbf{e}_9$	$\mathbf{e}_8$	$-\mathbf{e}_{15}$	$\mathbf{e}_{14}$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{14}$	$\mathbf{e}_{15}$	$\mathbf{e}_8$	$-\mathbf{e}_9$	$-\mathbf{e}_{10}$	$-\mathbf{e}_{11}$
$\mathbf{e}_{13}$	$\mathbf{e}_{13}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{14}$	$\mathbf{e}_9$	$\mathbf{e}_8$	$\mathbf{e}_{11}$	$-\mathbf{e}_{10}$
$\mathbf{e}_{14}$	$\mathbf{e}_{14}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{10}$	$-\mathbf{e}_{11}$	$\mathbf{e}_8$	$\mathbf{e}_9$
$\mathbf{e}_{15}$	$\mathbf{e}_{15}$	$\mathbf{e}_{14}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{11}$	$\mathbf{e}_{10}$	$-\mathbf{e}_9$	$\mathbf{e}_8$

Below is the list of 84 sets of zero divisors $\{\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d\}$ where $(\mathbf{e}_a + \mathbf{e}_b) \circ (\mathbf{e}_c + \mathbf{e}_d) = 0$:

Table A.2: Multiplication Table for $\mathbf{e}_i\mathbf{e}_j \in \mathbb{S}$

$\mathbf{e}_i\mathbf{e}_j$	$\mathbf{e}_8$	$\mathbf{e}_9$	$\mathbf{e}_{10}$	$\mathbf{e}_{11}$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{14}$	$\mathbf{e}_{15}$
$\mathbf{e}_0$	$\mathbf{e}_8$	$\mathbf{e}_9$	$\mathbf{e}_{10}$	$\mathbf{e}_{11}$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{14}$	$\mathbf{e}_{15}$
$\mathbf{e}_1$	$\mathbf{e}_9$	$-\mathbf{e}_8$	$-\mathbf{e}_{11}$	$\mathbf{e}_{10}$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{14}$
$\mathbf{e}_2$	$\mathbf{e}_{10}$	$\mathbf{e}_{11}$	$-\mathbf{e}_8$	$-\mathbf{e}_9$	$-\mathbf{e}_{14}$	$-\mathbf{e}_{15}$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$
$\mathbf{e}_3$	$\mathbf{e}_{11}$	$-\mathbf{e}_{10}$	$\mathbf{e}_9$	$-\mathbf{e}_8$	$-\mathbf{e}_{15}$	$\mathbf{e}_{14}$	$-\mathbf{e}_{13}$	$\mathbf{e}_{12}$
$\mathbf{e}_4$	$\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{14}$	$\mathbf{e}_{15}$	$-\mathbf{e}_8$	$-\mathbf{e}_9$	$-\mathbf{e}_{10}$	$-\mathbf{e}_{11}$
$\mathbf{e}_5$	$\mathbf{e}_{13}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{15}$	$-\mathbf{e}_{14}$	$\mathbf{e}_9$	$-\mathbf{e}_8$	$\mathbf{e}_{11}$	$-\mathbf{e}_{10}$
$\mathbf{e}_6$	$\mathbf{e}_{14}$	$-\mathbf{e}_{15}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{13}$	$\mathbf{e}_{10}$	$-\mathbf{e}_{11}$	$-\mathbf{e}_8$	$-\mathbf{e}_{10}$
$\mathbf{e}_7$	$\mathbf{e}_{15}$	$\mathbf{e}_{14}$	$-\mathbf{e}_{13}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{11}$	$\mathbf{e}_{10}$	$-\mathbf{e}_9$	$-\mathbf{e}_8$
$\mathbf{e}_8$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_9$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_5$	$\mathbf{e}_4$	$\mathbf{e}_7$	$-\mathbf{e}_6$
$\mathbf{e}_{10}$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$\mathbf{e}_4$	$\mathbf{e}_5$
$\mathbf{e}_{11}$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_5$	$\mathbf{e}_4$
$\mathbf{e}_{12}$	$-\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$
$\mathbf{e}_{13}$	$-\mathbf{e}_5$	$-\mathbf{e}_4$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{e}_{14}$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$
$\mathbf{e}_{15}$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_5$	$-\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$

Table A.3: $(\mathbf{e}_a+\mathbf{e}_b)\circ(\mathbf{e}_c+\mathbf{e}_d)=0$

$\{\mathbf{e}_1,\mathbf{e}_{10},\mathbf{e}_5,\mathbf{e}_{14}\}$	$\{\mathbf{e}_1,\mathbf{e}_{10},\mathbf{e}_4,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_1,\mathbf{e}_{10},\mathbf{e}_7,\mathbf{e}_{12}\}$	$\{\mathbf{e}_1,\mathbf{e}_{10},\mathbf{e}_6,-\mathbf{e}_{13}\}$
$\{\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_4,\mathbf{e}_{14}\}$	$\{\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_6,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_5,\mathbf{e}_{15}\}$	$\{\mathbf{e}_1,\mathbf{e}_{11},\mathbf{e}_7,-\mathbf{e}_{13}\}$
$\{\mathbf{e}_1,\mathbf{e}_{12},\mathbf{e}_2,\mathbf{e}_{15}\}$	$\{\mathbf{e}_1,\mathbf{e}_{12},\mathbf{e}_3,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_1,\mathbf{e}_{12},\mathbf{e}_6,\mathbf{e}_{11}\}$	$\{\mathbf{e}_1,\mathbf{e}_{12},\mathbf{e}_7,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_1,\mathbf{e}_{13},\mathbf{e}_6,\mathbf{e}_{10}\}$	$\{\mathbf{e}_1,\mathbf{e}_{13},\mathbf{e}_2,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_1,\mathbf{e}_{13},\mathbf{e}_7,\mathbf{e}_{11}\}$	$\{\mathbf{e}_1,\mathbf{e}_{13},\mathbf{e}_3,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_1,\mathbf{e}_{14},\mathbf{e}_2,\mathbf{e}_{13}\}$	$\{\mathbf{e}_1,\mathbf{e}_{14},\mathbf{e}_4,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_1,\mathbf{e}_{14},\mathbf{e}_3,\mathbf{e}_{12}\}$	$\{\mathbf{e}_1,\mathbf{e}_{14},\mathbf{e}_5,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_1,\mathbf{e}_{15},\mathbf{e}_3,\mathbf{e}_{13}\}$	$\{\mathbf{e}_1,\mathbf{e}_{15},\mathbf{e}_2,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_1,\mathbf{e}_{15},\mathbf{e}_4,\mathbf{e}_{10}\}$	$\{\mathbf{e}_1,\mathbf{e}_{15},\mathbf{e}_5,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_2,\mathbf{e}_9,\mathbf{e}_4,\mathbf{e}_{15}\}$	$\{\mathbf{e}_2,\mathbf{e}_9,\mathbf{e}_5,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_2,\mathbf{e}_9,\mathbf{e}_6,\mathbf{e}_{13}\}$	$\{\mathbf{e}_2,\mathbf{e}_9,\mathbf{e}_7,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_2,\mathbf{e}_{11},\mathbf{e}_5,\mathbf{e}_{12}\}$	$\{\mathbf{e}_2,\mathbf{e}_{11},\mathbf{e}_4,-\mathbf{e}_{13}\}$
$\{\mathbf{e}_2,\mathbf{e}_{11},\mathbf{e}_6,\mathbf{e}_{15}\}$	$\{\mathbf{e}_2,\mathbf{e}_{11},\mathbf{e}_7,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_2,\mathbf{e}_{12},\mathbf{e}_3,\mathbf{e}_{13}\}$	$\{\mathbf{e}_2,\mathbf{e}_{12},\mathbf{e}_5,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_2,\mathbf{e}_{12},\mathbf{e}_7,\mathbf{e}_9\}$	$\{\mathbf{e}_2,\mathbf{e}_{13},\mathbf{e}_3,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_2,\mathbf{e}_{13},\mathbf{e}_4,\mathbf{e}_{11}\}$	$\{\mathbf{e}_2,\mathbf{e}_{13},\mathbf{e}_6,-\mathbf{e}_9\}$
$\{\mathbf{e}_2,\mathbf{e}_{14},\mathbf{e}_5,\mathbf{e}_9\}$	$\{\mathbf{e}_2,\mathbf{e}_{14},\mathbf{e}_3,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_2,\mathbf{e}_{14},\mathbf{e}_7,\mathbf{e}_{11}\}$	$\{\mathbf{e}_2,\mathbf{e}_{15},\mathbf{e}_4,-\mathbf{e}_9\}$
$\{\mathbf{e}_2,\mathbf{e}_{15},\mathbf{e}_3,\mathbf{e}_{14}\}$	$\{\mathbf{e}_2,\mathbf{e}_{15},\mathbf{e}_6,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_3,\mathbf{e}_9,\mathbf{e}_6,\mathbf{e}_{12}\}$	$\{\mathbf{e}_3,\mathbf{e}_9,\mathbf{e}_4,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_3,\mathbf{e}_9,\mathbf{e}_7,\mathbf{e}_{13}\}$	$\{\mathbf{e}_3,\mathbf{e}_9,\mathbf{e}_5,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_3,\mathbf{e}_{10},\mathbf{e}_4,\mathbf{e}_{13}\}$	$\{\mathbf{e}_3,\mathbf{e}_{10},\mathbf{e}_5,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_3,\mathbf{e}_{10},\mathbf{e}_7,\mathbf{e}_{14}\}$	$\{\mathbf{e}_3,\mathbf{e}_{10},\mathbf{e}_6,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_3,\mathbf{e}_{12},\mathbf{e}_5,\mathbf{e}_{10}\}$	$\{\mathbf{e}_3,\mathbf{e}_{12},\mathbf{e}_6,-\mathbf{e}_9\}$
$\{\mathbf{e}_3,\mathbf{e}_{14},\mathbf{e}_4,\mathbf{e}_9\}$	$\{\mathbf{e}_3,\mathbf{e}_{13},\mathbf{e}_4,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_3,\mathbf{e}_{15},\mathbf{e}_5,\mathbf{e}_9\}$	$\{\mathbf{e}_3,\mathbf{e}_{13},\mathbf{e}_7,-\mathbf{e}_9\}$
$\{\mathbf{e}_3,\mathbf{e}_{15},\mathbf{e}_6,\mathbf{e}_{10}\}$	$\{\mathbf{e}_3,\mathbf{e}_{14},\mathbf{e}_7,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_4,\mathbf{e}_9,\mathbf{e}_7,\mathbf{e}_{10}\}$	$\{\mathbf{e}_4,\mathbf{e}_9,\mathbf{e}_6,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_4,\mathbf{e}_{10},\mathbf{e}_5,\mathbf{e}_{11}\}$	$\{\mathbf{e}_4,\mathbf{e}_{10},\mathbf{e}_7,-\mathbf{e}_9\}$
$\{\mathbf{e}_4,\mathbf{e}_{11},\mathbf{e}_6,\mathbf{e}_9\}$	$\{\mathbf{e}_4,\mathbf{e}_{11},\mathbf{e}_5,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_4,\mathbf{e}_{13},\mathbf{e}_6,\mathbf{e}_{15}\}$	$\{\mathbf{e}_4,\mathbf{e}_{13},\mathbf{e}_7,-\mathbf{e}_{14}\}$
$\{\mathbf{e}_4,\mathbf{e}_{14},\mathbf{e}_7,\mathbf{e}_{13}\}$	$\{\mathbf{e}_4,\mathbf{e}_{14},\mathbf{e}_5,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_4,\mathbf{e}_{15},\mathbf{e}_5,\mathbf{e}_{14}\}$	$\{\mathbf{e}_4,\mathbf{e}_{15},\mathbf{e}_6,-\mathbf{e}_{13}\}$
$\{\mathbf{e}_5,\mathbf{e}_{10},\mathbf{e}_6,\mathbf{e}_9\}$	$\{\mathbf{e}_5,\mathbf{e}_9,\mathbf{e}_6,-\mathbf{e}_{10}\}$
$\{\mathbf{e}_5,\mathbf{e}_{11},\mathbf{e}_7,\mathbf{e}_9\}$	$\{\mathbf{e}_5,\mathbf{e}_9,\mathbf{e}_7,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_5,\mathbf{e}_{12},\mathbf{e}_7,\mathbf{e}_{14}\}$	$\{\mathbf{e}_5,\mathbf{e}_{12},\mathbf{e}_6,-\mathbf{e}_{15}\}$
$\{\mathbf{e}_5,\mathbf{e}_{15},\mathbf{e}_6,\mathbf{e}_{12}\}$	$\{\mathbf{e}_5,\mathbf{e}_{14},\mathbf{e}_7,-\mathbf{e}_{12}\}$
$\{\mathbf{e}_6,\mathbf{e}_{11},\mathbf{e}_7,\mathbf{e}_{10}\}$	$\{\mathbf{e}_6,\mathbf{e}_{10},\mathbf{e}_7,-\mathbf{e}_{11}\}$
$\{\mathbf{e}_6,\mathbf{e}_{13},\mathbf{e}_7,\mathbf{e}_{12}\}$	$\{\mathbf{e}_6,\mathbf{e}_{12},\mathbf{e}_7,-\mathbf{e}_{13}\}$

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